On the Generalized Rumour Model in Continuous Time

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Abstract
In this paper, the Maki-Thompson model and the stochastic Daley-Kendall model are generalized using the probability approach for dynamics of spreading of rumour in a continuous time. New generalization model depends on a parameter representing the probability of the outcome of interaction between the two spreaders. The process of spreading rumors is described by a system of linear differential equations. The general solution for dynamics of spreading rumors is obtained and the dependence of the solution on the parameter is investigated numerically.

Mathematics Subject Classification: 90C30

Keywords: rumour model, Maki-Thompson model, Daley-Kendall model, system of linear differential equations
1 Introduction

Rumour may be regarded as either information or news. A rumour is not verified information and is always unconfirmed whereas news is always confirmed. It has been found that people generate rumours in an attempt to make sense of ambiguous, uncertain, or confusing situations [1, 2, 3, 4]. Rumours play a very important role in social life and have existed as a social fact since ancient times. The spreading of rumours is a social phenomenon and plays a significant role in a daily life. The rumour model explains the spreading of rumours and serves as a tool for understanding this social phenomenon. Rumours in economics have become more intensively discussed and investigated in last decades [1]. There are examples of dynamics based on communication and exchange at auctions, at the stock markets and trades. These backgrounds and motivations give basis for mathematical models of spreading of rumours [1, 5, 6]. The Maki - Thompson model [5] and the stochastic Daley - Kendall model [6] are the classical models of spreading of a single rumour. The Maki - Thompson model was slightly refined in a continuous time and a new general solution was obtained for dynamics of spreading of a rumour [1]. In this paper, the Maki - Thompson model and the stochastic Daley - Kendall model are generalized using the probability approach to the interaction of two spreaders who are spreading the rumour among the population. This article continues exploration of non-traditional tasks of control theory some of which were discussed earlier in [7, 8, 9,10]. The study issues are close to the problems studied in the theory of networking games [11].

2 The main results and generalized model.

In this paper, which is based on the article [1], we consider three subpopulations: Ignorants who have not heard the rumour yet, spreaders who know the rumour and spread the rumour among the population, stiflers who as former spreaders knew the rumour but either forget the rumour or give up to spread the rumour among population.

Table 1 Three types of interactions in the rumour process

<table>
<thead>
<tr>
<th>$S \rightarrow I$</th>
<th>Daley – Kendall</th>
<th>Maki – Thompson</th>
<th>$S \rightarrow R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i_k = i_{k-1} - 1$</td>
<td>$i_k = i_{k-1}$</td>
<td>$i_k = i_{k-1}$</td>
<td>$i_k = i_{k-1}$</td>
</tr>
<tr>
<td>$s_k = s_{k-1} + 1$</td>
<td>$s_k = s_{k-1} - 2$</td>
<td>$s_k = s_{k-1} - 1$</td>
<td>$s_k = s_{k-1} - 1$</td>
</tr>
<tr>
<td>$r_k = r_{k-1}$</td>
<td>$r_k = r_{k-1} + 2$</td>
<td>$r_k = r_{k-1} + 1$</td>
<td>$r_k = r_{k-1} + 1$</td>
</tr>
<tr>
<td>$\triangle i_k = i_k - i_{k-1} = -1$</td>
<td>$\triangle i_k = 0$</td>
<td>$\triangle i_k = 0$</td>
<td>$\triangle i_k = 0$</td>
</tr>
<tr>
<td>$\triangle s_k = s_k - s_{k-1} = 1$</td>
<td>$\triangle s_k = -2$</td>
<td>$\triangle s_k = 1$</td>
<td>$\triangle s_k = 1$</td>
</tr>
<tr>
<td>$\triangle r_k = 0$</td>
<td>$\triangle r_k = 2$</td>
<td>$\triangle r_k = 1$</td>
<td>$\triangle r_k = 1$</td>
</tr>
</tbody>
</table>

The model is defined by the following transition probability scheme:
On the generalized rumor model in continuous time

<table>
<thead>
<tr>
<th>Interactions</th>
<th>Transitions</th>
<th>Transition probabilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S \rightarrow I$</td>
<td>${(i - 1, s + 1, r)}$</td>
<td>$\frac{1}{n}$</td>
</tr>
<tr>
<td>$S \rightarrow S$</td>
<td>${(i, s - 1, r + 1), p, \frac{s-1}{n}} \cup {(i, s - 1, r + 1), (1-p)\frac{s-1}{n}}$</td>
<td>$\frac{s-1}{n}$</td>
</tr>
<tr>
<td>$S \rightarrow R$</td>
<td>$(i, s - 2, r + 2)$</td>
<td>$\frac{r}{n}$</td>
</tr>
</tbody>
</table>

The total population size is given by

$$i + s + r = n + 1 = N$$  \hspace{1cm} (1)

We introduce the initial conditions

$$i(0) = \alpha \cdot n, \quad s(0) = \beta \cdot n, \quad r(0) = \gamma \cdot n, \quad \alpha, \beta, \gamma \in (0, 1), \quad \alpha + \beta + \gamma = 1, \quad \beta \cdot n \geq 1$$

The conditional expected value of $\Delta i_k$ is estimated by

$$\Delta i_k = (-1) \frac{i_{k-1}}{n} \Delta t \Rightarrow \frac{\Delta i_k}{\Delta t} = (-1) \frac{i_{k-1}}{n} \quad \Delta t \to 0.$$  

Then we get

$$\frac{di}{dt} = (-1) \frac{i(t)}{n} \Rightarrow i(t) = Ce^{-\frac{t}{n}}, \quad C = i(0) = \alpha \cdot n, \quad i(t) = \alpha \cdot n \cdot e^{-\frac{t}{n}}.$$  

The conditional expected value of $\Delta s_k$, taking into account the relation (1), for $p \in [0, 1]$, is estimated by

$$\frac{\Delta s_k}{\Delta t} = \frac{i_{k-1}}{n} \cdot \frac{n - i_{k-1}}{n} - (1 - p) \cdot \frac{s_{k-1} - 1}{n}, \quad \Delta t \to 0,$$

then we get

$$\frac{ds}{dt} = 2 \cdot \frac{i(t)}{n} \cdot \frac{1 - p}{n} \cdot s(t) + \frac{1 - p - n}{n},$$

so we get

$$\frac{ds}{dt} + \frac{1 - p}{n} \cdot s(t) = 2 \cdot \alpha \cdot e^{-\frac{t}{n}} + \frac{1 - p - n}{n}$$  \hspace{1cm} (2)

The conditional expected value of $\Delta r_k$, for $p \in [0, 1]$, is estimated by

$$\Delta r_k = (1 - p) \frac{s_{k-1} - 1}{n} \Delta t + \frac{s_{k-1} + r_{k-1} - 1}{n} \Delta t, \quad \Delta t \to 0.$$  

Then we get

$$\frac{dr}{dt} = (-1) \frac{i(t)}{n} + (1 - p) \cdot \frac{s(t) - 1}{n} + 1.$$
The system of equations

\[
\begin{align*}
\frac{di}{dt} &= (-1)^i \frac{i(t)}{n} \\
\frac{ds}{dt} &= 2 \cdot \frac{i(t)}{n} - \frac{1-p}{n} \cdot s(t) + \frac{1-p-n}{n} \\
\frac{dr}{dt} &= (-1)^i \frac{i(t)}{n} + (1-p) \cdot \frac{s(t)}{n} + 1
\end{align*}
\]

(3)

generalizes Maki-Thompson model and Daley-Kendall model.

We solve the equation (2) by method Bernoulli [12], replacing a variable in the equation

\[ s = u \cdot \nu, \quad \dot{s} = \dot{u} \cdot \nu + u \cdot \dot{\nu}. \]

We have the system of equations

\[
\begin{align*}
\dot{\nu} + \frac{1-p}{n} \cdot \nu &= 0 \\
\dot{u} \cdot \nu &= 2 \cdot \alpha \cdot e^{-\frac{t}{n}} + \frac{1-p-n}{n}
\end{align*}
\]

From the first equation we find the function \( \nu \)

\[
\frac{d\nu}{dt} = -\frac{1-p}{n} \cdot \nu \quad \rightarrow \quad \int \frac{d\nu}{\nu} = -\frac{(1-p)}{n} \int dt \rightarrow \ln |\nu| = -\frac{1-p}{n} \cdot t \rightarrow \nu = \exp\left(-\frac{(1-p)}{n} \cdot t\right)
\]

From the second equation we get

\[
\frac{du}{dt} = 2 \cdot \alpha \cdot e^{-\frac{t}{n}} + \frac{1-p-n}{n} \cdot e^{\frac{(1-p) \cdot t}{n}};
\]

then we have

\[ u = \frac{(1-p-n)}{(1-p)} \cdot e^{\frac{(1-p) \cdot t}{n}} - \frac{2 \cdot \alpha \cdot n}{p} \cdot e^{-\frac{t}{n}} + C. \]

The general solution for \( s(t) \) and \( p \in (0,1) \) has the form

\[ s(t) = \frac{(1-p-n)}{(1-p)} - \frac{2 \cdot \alpha \cdot n}{p} \cdot e^{-\frac{t}{n}} + \left(\beta \cdot n - \frac{(1-p-n)}{(1-p)} + \frac{2 \cdot \alpha \cdot n}{p}\right) \cdot e^{-\frac{(1-p) \cdot t}{n}}. \]

The general solution for \( s(t) \) and \( p = 0 \) has the form

\[ s(t) = (1-n) - [2 \cdot \alpha \cdot t - 1 - n(\beta - 1)] \cdot e^{-\frac{t}{n}}. \]

The general solution for \( s(t) \) and \( p = 1 \) has the form

\[ s(t) = (\beta + 2 \cdot \alpha) \cdot n - 2 \cdot \alpha \cdot n \cdot e^{-\frac{t}{n}} - t. \]

The general solution for \( r(t) \) and \( p \in (0,1) \) has the form

\[ r(t) = \gamma \cdot n + n \cdot \left[ \frac{(1-p-n)}{n} - \frac{2 \cdot \alpha \cdot (1-p)}{p} - \alpha \right] + n \cdot \left( \beta - \frac{(1-p-n)}{(1-p) \cdot n} + \frac{2 \cdot \alpha}{p} \right) - \]
The general solution for $r(t)$ and $p = 0$ has the form

$$r(t) = 2\cdot n \cdot e^{-\frac{t}{n}} \left(\frac{1}{n} + 1\right) - \alpha \cdot n \cdot e^{-\frac{t}{n}} \left(\frac{1}{n} - \alpha\right) - \alpha \cdot n - 2\cdot n + \gamma \cdot n.$$}

The general solution for $r(t)$ and $p = 1$ has the form

$$r(t) = t + \alpha \cdot n \cdot e^{-\frac{t}{n}} - \alpha \cdot n + \gamma \cdot n.$$}

For example, consider several options for the behavior of $s(t)$ depending on the $p \in (0, 1)$:

*Fig. 1. Graph function $s(t)$, $n = 200$, $\alpha = 0.9$, $\beta = 0.1$*

The point $T$ where $s(T) = 0$ is the time moment of finish of the rumour spreading. We can consider the rumour process on the time interval $(0, T)$.

Dynamics of changes in the number of ignorants are illustrated in Fig. 2.
3 Conclusion

In this paper, a new generalized model for dynamics of rumour spreading in a continuous time is suggested. The model generalizes the classical Maki-Thompson model for continuous time [1] and the classical stochastic Da-ley-Kendall model on the basis of a probability approach for the outcome of the interaction between the two spreaders. Our model may serve as a tool understanding the social phenomenon of rumour better. When the rumour process stops, we have $s(T) = 0$. We give the numerical results for $p = 0.3, p = 0.5, p = 0.7$ and present them in Fig.1. We obtain a new general solution in continuous time for dynamics of spreading of the rumour in the rumour process.

Acknowledgements. This research is supported by Saint Petersburg University (grant 9.38.205.2014).

References

http://dx.doi.org/10.1007/s10100-009-0120-4


Received: August 30, 2016; Published: October 16, 2016