A Note on Renewal Theory for $T$-iid Random Fuzzy Variables

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Abstract

In this note, we investigate a classical version of renewal theories in the $T$-independent and identically distributed random fuzzy variables. For special cases, we consider the case for $T = \text{min}$ and $T = \text{Archimedean}$ $t$-norm.

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1 Introduction and preliminaries

A number of studies [1-9] have investigated renewal theory in the random fuzzy environment based on the concept of fuzzy variable and random fuzzy variable. Recently, Hong [1] investigated renewal theories in the $T$-independent random fuzzy environment based on the concept of random fuzzy variable including the cases for $T = \text{min}$ and $T = \text{Archimedean}$ $t$-norm. In this note, we consider a classical version of renewal theories in the $T$-iid random fuzzy variables. For special cases, we consider the case for $T = \text{min}$ and $T = \text{Archimedean}$ $t$-norm.

For basic notations and definitions for fuzzy variables and their $T$-norm based operations, please refer to the paper [1].

A random fuzzy variable [6] is a function from a possibility space $(\Theta, \mathcal{P}(\Theta), \text{Pos})$ to a collection of random variables $\mathcal{F}$. The expected value of random fuzzy
variable is defined by Liu and Liu [4] as
\[
E[\xi] = \int_{0}^{\infty} Cr\{\theta \in \Theta | E[\xi(\theta)] \geq r\}dr - \int_{-\infty}^{0} Cr\{\theta \in \Theta | E[\xi(\theta)] \leq r\}dr.
\]

**Definition 1** [2]. Random fuzzy variables \(\xi_1, \xi_2, \cdots, \xi_n\) are said to be \(T\)-independent if
(a) \(\xi_1(\theta), \xi_2(\theta), \cdots, \xi_n(\theta)\) are independent random variables for each \(\theta\);
(b) \(E[\xi_1(\cdot)], E[\xi_2(\cdot)], \cdots, E[\xi_n(\cdot)]\) are \(T\)-independent fuzzy variables.

It is noted that for a random fuzzy variables \(\xi\) and a Borel set \(B\) of \(R\), \(P\{\xi(\cdot) \in B\}\) is a fuzzy variable.

**Definition 2** [2]. The random fuzzy variables \(\xi\) and \(\eta\) are said to be identically distributed if for any element \(B\) of Borel field \(\mathcal{B}\) of \(R\), \(P\{\xi(\cdot) \in B\}\) and \(P\{\eta(\cdot) \in B\}\) are identically distributed fuzzy variables.

Let \(\Theta\) be a family of probability distribution functions on \(R\) and let \((\Theta, \mathcal{P}(\Theta), Pos)\) be a possibility space and \(\mathcal{F}\) be a family of distributions of random variables. Let \(\xi : \Theta \rightarrow \mathcal{F}\) be a random fuzzy variable. We denote by \(\Theta_1^\infty = \Pi_{n=1}^\infty \Theta\) the space consisting of all infinite sequences of probability distribution functions \((\theta_1, \theta_2, \cdots), \theta_n \in \Theta\) and \(R_1^\infty = \Pi_{i=1}^\infty R\) the space consisting of all infinite sequences \((x_1, x_2, \cdots)\) of real numbers. We take \(B_1^\infty\) to be the Borel \(\sigma\)-field of \(R_1^\infty\). Define a possibility measure \(Pos_1^\infty\) on \(\Theta_1^\infty\) such that for any \(A \subset \Theta_1^\infty\),
\[
Pos_1^\infty\{A\} = \sup_{(\theta_1, \theta_2, \cdots) \in A} T(Pos\{\theta_1\}, Pos\{\theta_2\}, \cdots).
\]
Then \((\Theta_1^\infty, \mathcal{P}(\Theta_1^\infty), Pos_1^\infty)\) is called the \(T\)-product possibility measure of \((\theta_1, \theta_2, \cdots)\).

Let \(P_\theta\) be the probability measure on \(R\) with probability distribution \(\theta\). For each \(\bar{\theta} = (\theta_1, \theta_2, \cdots)\) define a probability measure on \((R_1^\infty, B_1^\infty)\) so that \(P_{\bar{\theta}} = \Pi_{i=1}^\infty P_\theta\), the product probability measure of \(P_\theta, i = 1, 2, \cdots\). Define a process \(\{X_n\}\) on \((R_1^\infty, B_1^\infty)\) such that \(X_n(x_1, x_2, \cdots) = x_n\). By the definition of \(P_{\bar{\theta}}\), the process \(\{X_n\}\) is independent with respect to \(P_{\bar{\theta}}\) and \(\theta_n\) is the probability distribution of \(X_n\). We now define a random fuzzy variables \(\{\xi_n\}\) on \((\Theta_1^\infty, \mathcal{P}(\Theta_1^\infty), Pos_1^\infty)\) such that \(\xi_n(\bar{\theta}) = X_n\) with respect to \(P_{\bar{\theta}}\) and set \(S_0 = 0, S_n = \xi_1 + \xi_2 + \cdots + \xi_n, n = 1, 2, \cdots\). Then, by Theorem 2 [2], the random fuzzy variables \(\xi_n, n = 1, 2, \cdots\) on \((R_1^\infty, B_1^\infty)\) are \(T\)-iid random fuzzy variables and identically distributed with a random fuzzy variable \(\xi\).

## 2 Random fuzzy renewal theories

From this section, we additionally assume that \(\Theta\) is a set of probability distribution functions such that \(\theta(0) = 0, \theta(0) < 1\). Let \(\xi_n\) denotes the time between
the $(n-1)$th and the $n$th events, known as the inter-arrival times, $n = 1, 2, \cdots$, respectively. Define
\[
S_0 = 0, \quad S_n = \xi_1 + \xi_2 + \cdots + \xi_n, \quad n \geq 1,
\]

If the inter-arrival times $\xi_n$, $n = 1, 2, \cdots$ are random fuzzy variables then the process $\{S_n, \ n \geq 1\}$ is called a random fuzzy renewal process.

Let $N(t)$ denotes the total number of the events that have occurred by time $t$. Then we have
\[
N(t) = \max\{n | 0 < S_n \leq t\}.
\]

For any fixed $\bar{\theta} = (\theta_1, \theta_2, \cdots) \in \Theta^\infty_1$, it is clear that $N(t)(\bar{\theta})$ is a random variable with the probability distribution $P\{N(t)(\bar{\theta}) = n\} = P\{S_n(\bar{\theta}) \leq t\} - P\{S_{n+1}(\bar{\theta}) \leq t\}, n = 1, 2, \cdots,$
where $S_n(\bar{\theta}) = \sum_{i=1}^n \xi_i(\bar{\theta}) = \sum_{i=1}^n X_i$ w.r.t. $P_{\bar{\theta}}$. We call $N(t)$ the random fuzzy renewal variable.

For each $\bar{\theta} \in \Theta^\infty_1$, $E[N(t)(\bar{\theta})]$ is the expected values of the random variables $N(t)(\bar{\theta})$. However, when $\bar{\theta}$ is varied all over in $\Theta^\infty_1$, $E[N(t)(\bar{\theta})]$, as a function of $\bar{\theta} \in \Theta^\infty_1$, is fuzzy variable and their $\alpha$-pessimistic and $\alpha$-optimistic values can be expressed by
\[
E[N(t)(\bar{\theta})]'_\alpha = \inf\{t \mid \mu_{E[N(t)(\bar{\theta})]}(t) \geq \alpha\},
E[N(t)(\bar{\theta})]''_\alpha = \sup\{t \mid \mu_{E[N(t)(\bar{\theta})]}(t) \geq \alpha\}.
\]

Recently, Hong [1] investigated random fuzzy elementary renewal theories for $T$-iid random fuzzy variables as follows.

**Theorem 1 [1]**. Let $\{\xi_n\}$ be a $T$-iid random fuzzy process on $(\Theta^\infty_1, \mathcal{P}(\Theta^\infty_1), Pos_1^\infty)$ such that $\|E[\xi_1(\bar{\theta})]_\alpha\| < \infty$, $\alpha \in (0, 1]$. Then we have, for $\alpha \in (0, 1]$,
\[
d_H\left(\frac{E[N(t)(\bar{\theta})]_\alpha}{t}, \left[\frac{1}{KE[\xi_1(\bar{\theta})]}\right]_\alpha\right) \to 0 \quad \text{as} \quad t \to \infty.
\]

**Corollary 1 [1]**. Let $\{\xi_n\}$ be a $T$-iid random fuzzy process on $(\Theta^\infty_1, \mathcal{P}(\Theta^\infty_1), Pos_1^\infty)$. Suppose that $\|E[\xi_1(\bar{\theta})]_\alpha\| < \infty$, $\alpha \in (0, 1]$ and $T$ is an Archimedean $t$-norm, then we have, for all $0 < \alpha \leq 1$
\[
d_H\left(\frac{E[N(t)(\bar{\theta})]_\alpha}{t}, \frac{1}{E[\xi_1(\bar{\theta})]_1}\right) \to 0 \quad \text{as} \quad t \to \infty.
\]

**Corollary 2 [1]**. Let $\{\xi_n\}$ be a $T$-iid random fuzzy process on $(\Theta^\infty_1, \mathcal{P}(\Theta^\infty_1), Pos_1^\infty)$. Suppose that $\|E[\xi_1(\bar{\theta})]_\alpha\| < \infty$, $\alpha \in (0, 1]$ and $T = min$, then we have, for all $0 < \alpha \leq 1$
\[
d_H\left(\frac{E[N(t)(\bar{\theta})]_\alpha}{t}, \left[\frac{1}{E[\xi_1(\bar{\theta})]}\right]_\alpha\right) \to 0 \quad \text{as} \quad t \to \infty.
\]
A scale density is a density of the form
\[ \sigma^{-1} f \left( \frac{x}{\sigma} \right) \]
where \( \sigma > 0 \). The parameter \( \sigma \) is called a scale parameter.

The following lemma is easy to check.

**Lemma 1.** Let \( f \) be a density function with \( E_f = \int_x x f(x) dx < \infty \). Let \( \theta^\sigma(x) = \int_x^{-\infty} \sigma^{-1} f \left( \frac{y}{\sigma} \right) dy \), then

\[ 1 - \theta^\sigma(\sigma a) = 1 - \theta^1(a) \]

and

\[ \int x d\theta^\sigma(x) = \sigma \int x f(x) dx. \]

**Example 1.** The followings are examples of scale densities.

Normal density \( N(0, \sigma^2) \):

\[ f(x|\sigma^2) = \frac{1}{(2\pi)^{1/2}\sigma} e^{-x^2/2\sigma^2}, \]

Gamma density \( \Gamma(\alpha, \beta) \) (\( \alpha \) fixed):

\[ f(x|\alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} y^{\alpha-1} e^{-y/\beta} I_{(0,\infty)}(x), \]

Uniform density \( U(0, \beta) \):

\[ f(x|\beta) = \frac{1}{\beta} I_{(0,\beta)}(x) \]

\( t \)-distribution with \( \alpha \) degree of freedom \( T(\alpha, \sigma^2) \) (\( \alpha > 0 \) fixed):

\[ f(x|\alpha, \sigma^2) = \frac{\Gamma[(\alpha + 1)/2]}{\sigma(\alpha\pi)^{1/2}} \left( 1 + \frac{x^2}{\alpha\sigma^2} \right)^{-(\alpha+1)/2} \]

It is noted that a class of scale densities is a totally ordered set with the
stochastic ordering.

In the next result, we assume that \( \Theta \) is a class of scale densities and
\( \mu_{E[\xi_1(\theta)]}(t) \) is a fuzzy number, and consider classical version of renewal theo-
ries for \( T \)-iid random fuzzy variables.

**Theorem 2.** Let \( \Theta = \{\theta^\sigma|0 < \sigma < \infty\} \) be a class of scale densities of \( f \). Let \( \{\xi_n\} \) be a \( T \)-iid random fuzzy process on \((\Theta_1^\infty, \mathcal{P}(\Theta_1^\infty), Pos_1^\infty)\). If \( E[||E[\xi_1(\theta)]||] < \infty \), then we have

\[ \lim_{t\to\infty} \frac{E[N(t)]}{t} = E \left[ \frac{1}{K E[\xi_1(\theta)]]} \right]. \]
We need the following lemma.

**Lemma 2.** Let \( \Theta = \{ \theta^\sigma | 0 < \sigma < \infty \} \) be a class of scale densities of \( f \). Let \( \{ \xi_n \} \) be a \( T \)-i.i.d random fuzzy process on \( (\Theta_1^\infty, \mathcal{P}(\Theta_1^\infty), Pos_1^\infty) \). Then, we have

\[
\sup_{t>1} E[N(t)(\bar{\theta}'_\alpha)] \leq C \frac{1}{E[\xi_1(\bar{\theta})]'_\alpha}.
\]

for some constant \( C > 0 \).

**Proof** Since \( \mu_{E[\xi_1(\bar{\theta})]}(t) \) is fuzzy convex and upper semi continuous and \( \Theta \) is a totally ordered set with the stochastic ordering, for \( \alpha \in (0, 1] \) there exist \( \theta'_\alpha, \theta''_\alpha \in \Theta \) such that

\[
\{ \theta \in \Theta : Pos(\theta) \geq \alpha \} = \{ \theta_1 \in \Theta : \mu_{E[\xi_1(\bar{\theta})]}(t) \geq \alpha \} = \{ \theta \in \Theta : \theta'_\alpha \leq \theta \leq \theta''_\alpha \}.
\]

Then we clearly have

\[
E[N(t)(\bar{\theta}'_\alpha)] \leq [E[N(t)(\bar{\theta})]'_\alpha \leq [E[N(t)(\bar{\theta})]'_\alpha \leq E[N(t)(\bar{\theta}'_\alpha)]
\]

where \( \bar{\theta}''_\alpha = (\theta''_\alpha, \theta''_\alpha, \cdots) \) and \( \bar{\theta}'_\alpha = (\theta'_\alpha, \theta'_\alpha, \cdots) \). Let \( \theta'_\alpha = \theta f_1(\alpha), \theta''_\alpha = \theta f_2(\alpha) \). Then \( f_1(\alpha) \) is bounded increasing function and \( f_2(\alpha) \) is decreasing function such that \( f_1(\alpha) \leq f_2(\alpha) \), since \( \mu_{E[\xi_1(\bar{\theta})]}(t) \) is fuzzy convex. We also note that by Lemma 1,

\[
E[\xi_1(\bar{\theta})]'_\alpha = \int x d\theta f_1(\alpha)(x) = f_1(\alpha)E_f.
\]

We chose \( a > 0 \) such that \( 0 < \theta'_\alpha(a f_1(\alpha)) < 1 \) and let \( p_\alpha = 1 - \theta'_\alpha(a f_1(\alpha)) \). Define new inter-arrival times via truncation \( \tilde{X}_n^1 = a f_1(\alpha)I\{ X_n^1 > a f_1(\alpha) \} \). Thus \( \tilde{X}_n^1 = 0 \) with probability \( \theta'_\alpha(a f_1(\alpha)) \) and equals \( a f_1(\alpha) \) with probability \( 1 - \theta'_\alpha(a f_1(\alpha)) \). Let \( \tilde{N}(t)(\bar{\theta}'_\alpha) \) denote the counting process obtained by using these new inter-arrival times, it follows that \( N(t)(\bar{\theta}'_\alpha) \leq \tilde{N}(t)(\bar{\theta}'_\alpha), t > 0 \). Letting \( H_n(\bar{\theta}'_\alpha) \) denote the number of arrivals that occurs at time \( n a f_1(\alpha) \), we conclude that \( \{ H_n(\bar{\theta}'_\alpha) \} \) is i.i.d. with a geometric distribution with success probability \( p_\alpha \). Letting \([x]\) denote the smallest integer \( \geq x \), we have the inequality

\[
N(t)(\bar{\theta}'_\alpha) \leq \tilde{N}(t)(\bar{\theta}'_\alpha) \leq H(t)(\bar{\theta}'_\alpha) = \sum_{n=1}^{[t/a f_1(\alpha)]} H_n(\bar{\theta}'_\alpha), t > 0.
\]

Observing that \( p_\alpha = p_1 \) by Lemma 1, and

\[
E(H_t(\bar{\theta}'_\alpha)) = [t/a f_1(\alpha)]E(H_1(\bar{\theta}'_\alpha)) \leq \left( \frac{t}{a f_1(\alpha)} + 1 \right) \frac{1}{p_\alpha} = \frac{t + a f_1(\alpha)}{a f_1(\alpha)p_\alpha}.
\]
we obtain
\[
\frac{E[N(t)(\bar{\theta}_\alpha')]}{t} \leq \left( \frac{t + af_1(\alpha)}{taf_1(\alpha)} \right) \frac{1}{p_1}.
\]
Since \(0 < f_1(\alpha) \leq f_1(1)\), there exist constants \(C_1 > 0\) such that for \(t > 1\),
\[
t + af_1(\alpha) \leq t + af_1(1) = \frac{1}{af_1(\alpha)} + \frac{f_1(1)}{tf_1(\alpha)} \leq C_1 \frac{1}{f_1(\alpha)},
\]
and hence
\[
\frac{E[N(t)(\bar{\theta}_\alpha')]}{t} \leq \frac{C_1 E_f}{p_1} \frac{1}{E[\xi_1(\bar{\theta})]_\alpha'}.
\]

**Proof of Theorem 2.** We first note that
\[
\frac{E[N(t)]}{t} = \int_0^1 \frac{1}{2} \left( \frac{E[N(t)(\bar{\theta})']}{t} + \frac{E[N(t)(\bar{\theta})'']}{t} \right) d\alpha.
\]
and
\[
E \left[ \frac{1}{KE[\xi_1(\bar{\theta})]} \right] = \frac{1}{2} \int_0^1 \left( \left[ \frac{1}{KE[\xi_1(\bar{\theta})]} \right]'_\alpha + \left[ \frac{1}{KE[\xi_1(\bar{\theta})]} \right]''_\alpha \right) d\alpha.
\]
From Theorem 1, we have that, for \(\alpha \in (0, 1]\),
\[
\lim_{t \to \infty} \frac{E[N(t)(\bar{\theta})']}{t} = \left[ \frac{1}{KE[\xi_1(\bar{\theta})]} \right]''_\alpha
\]
and
\[
\lim_{t \to \infty} \frac{E[N(t)(\bar{\theta})'']}{t} = \left[ \frac{1}{KE[\xi_1(\bar{\theta})]} \right]'_\alpha.
\]
Hence, it suffices to prove that
\[
\lim_{t \to \infty} \int_0^1 \frac{E[N(t)(\bar{\theta})']}{t} d\alpha = \int_0^1 \left[ \frac{1}{KE[\xi_1(\bar{\theta})]} \right]'' d\alpha
\]
and
\[
\lim_{t \to \infty} \int_0^1 \frac{E[N(t)(\bar{\theta})'']}{t} d\alpha = \int_0^1 \left[ \frac{1}{KE[\xi_1(\bar{\theta})]} \right]' d\alpha.
\]
By Lemma 2, we have for \(t > 1\),
\[
\int_0^1 \left( \frac{E[N(t)(\bar{\theta})']}{t} \right) d\alpha \leq \int_0^1 \left( \frac{E[N(t)(\bar{\theta})'']}{t} \right) d\alpha \leq \int_0^1 \left( \frac{E[N(t)(\bar{\theta}_\alpha')]}{t} \right) d\alpha
\]
and
\[
\int_0^1 \left( \frac{E[N(t)(\bar{\theta}_\alpha')]}{t} \right) d\alpha \leq C \int_0^1 \frac{1}{E[\xi_1(\bar{\theta})]_\alpha'} d\alpha \leq CE \left\| \frac{1}{E[\xi_1(\bar{\theta})]} \right\|.
\]
Therefore, by Dominated Convergence Theorem, we immediately have the result.

If $T = \min$, we have the following result.

**Corollary 3.** Let $T = \min$ and $\Theta = \{\theta^\sigma|0 < \sigma < \infty\}$ be a class of scale densities of $f$. Let $\{\xi_n\}$ be a $T$-iid random fuzzy process on $(\Theta^\infty, \mathcal{P}(\Theta^\infty), Pos^\infty)$. If $E\|\frac{1}{E[\xi(\theta)]}\| < \infty$, then we have

$$\lim_{t \to \infty} \frac{E[N(t)]}{t} = E \left[ \frac{1}{E[\xi(\theta)]} \right].$$

If $T$ is an Archimedean $t$-norm, we have the following result.

**Corollary 4.** Let $T$ be an Archimedean $t$-norm and $\Theta = \{\theta^\sigma|0 < \sigma < \infty\}$ be a class of scale densities of $f$. Let $\{\xi_n\}$ be a $T$-iid random fuzzy process on $(\Theta^\infty, \mathcal{P}(\Theta^\infty), Pos^\infty)$. If $E\|\frac{1}{E[\xi(\theta)]}\| < \infty$, then we have

$$\lim_{t \to \infty} \frac{E[N(t)]}{t} = \frac{1}{2} \left( \frac{1}{E[\xi(\theta)]} + \frac{1}{E[\xi(\theta)]'} \right).$$

**Example 2.** Let $T = (\langle 0, 2/3, T_1 \rangle, \langle 2/3, 1, T_2 \rangle)$ and $\Theta = \{\theta^\sigma|0 < \sigma < \infty\}$ be a class of scale densities of exponential distribution with mean parameter $\sigma$ with

$$\mu_{E[\xi(\theta)]}(\sigma) = \begin{cases} \sigma^2 & \text{for } \sigma \in [0, 1], \\ 2 - \sigma & \text{for } \sigma \in [1, 2], \\ 0 & \text{otherwise}. \end{cases}$$

Then

$$E[\xi(\theta)]' = \sqrt{\alpha}, E[\xi(\theta)]'' = 2 - \alpha$$

and

$$\frac{1}{E[\xi(\theta)]'} = \frac{1}{2 - \alpha}, \frac{1}{E[\xi(\theta)]''} = \frac{1}{\sqrt{\alpha}}$$

Then

$$\mu_{\frac{1}{E[\xi(\theta)]}}(\sigma) = \begin{cases} \frac{2\sigma - 1}{\sigma^2} & \text{for } \sigma \in [1/2, 1], \\ \frac{1}{\sigma^2} & \text{for } \sigma \in [1, \infty), \\ 0 & \text{otherwise}. \end{cases}$$

We also have

$$[KE[\xi(\theta)]]' = \begin{cases} \sqrt{\frac{2}{3}} & \text{for } \alpha \in [0, 2/3], \\ 1 & \text{for } \alpha \in (2/3, 1], \end{cases}$$

$$[KE[\xi(\theta)]]'' = \begin{cases} \frac{4}{3} & \text{for } \alpha \in [0, 2/3], \\ 1 & \text{for } \alpha \in (2/3, 1]. \end{cases}$$
and hence
\[
\left[ \frac{1}{KE[\xi_1(\theta)]]_\alpha} \right]' = \frac{1}{[KE[\xi_1(\theta)]]]'_\alpha} = \begin{cases} 
\frac{3}{4} & \text{for } \alpha \in [0, 2/3], \\
1 & \text{for } \alpha \in (2/3, 1].
\end{cases}
\]
\[
\left[ \frac{1}{KE[\xi_1(\theta)]]_\alpha} \right]'' = \frac{1}{[KE[\xi_1(\theta)]]]''_\alpha} = \begin{cases} 
\sqrt{\frac{3}{2}} & \text{for } \alpha \in [0, 2/3], \\
1 & \text{for } \alpha \in (2/3, 1].
\end{cases}
\]

Then
\[
\mu_{\frac{1}{KE[\xi_1(\theta)]}}(\sigma) = \begin{cases} 
\frac{2}{3} & \text{for } \sigma = 1, \\
\frac{3}{4} & \text{for } \sigma \in [3/4, 1) \cup (1, \sqrt{3/2}], \\
0 & \text{otherwise}.
\end{cases}
\]

We have
\[
E \left[ \frac{1}{E[\xi_1(\theta)]]} \right] = \int_{0}^{1} \frac{1}{2} \left( \frac{1}{2 - \alpha} + \frac{1}{\sqrt{\alpha}} \right) d\alpha = \frac{1}{2} (2 + \log 2)
\]
and similarly,
\[
E \left[ \frac{1}{KE[\xi_1(\theta)]]_\alpha} \right] = \frac{1}{3} \left( \frac{7}{4} + \sqrt{\frac{3}{2}} \right)
\]

If \( T \) be an continuous Archimedean \( t \)-norm, then by Theorem 2
\[
\lim_{t \to \infty} \frac{E[N(t)]}{t} = \frac{1}{3} \left( \frac{7}{4} + \sqrt{\frac{3}{2}} \right).
\]

If \( T = \min \), then by Corollary 3
\[
\lim_{t \to \infty} \frac{E[N(t)]}{t} = \frac{1}{2} (2 + \log 2).
\]

If \( T \) be an Archimedean \( t \)-norm, then by Corollary 4
\[
\lim_{t \to \infty} \frac{E[N(t)]}{t} = 1.
\]

References


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