Toric Deformation of the Hankel Variety

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Abstract

A combinatorial criterium for detecting the normality of the semigroup under the toric deformation (initial algebra of the coordinate ring) of the Hankel projective variety is studied and applied. We discuss properties of the affine semigroup.

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Introduction

Hankel matrices, with entries in any field $K$, appear in operators theory, in algebraic geometry and in many other topics of algebraic and geometric modelizations. Since generic matrices theorize the Grassmannian varieties, that are classical objects in geometry, it is natural to study the Hankel variety, a subvariety of the Grassmannian variety $G(r,n)$, coming from generic Hankel matrices. In [6] the Hankel variety $H(r,n)$ of Hankel $r-$planes of $\mathbb{P}^n$ was introduced from the geometric point of view and $H(1,n)$, the variety of lines in $\mathbb{P}^n$, with related properties, was studied in [5],[7]. The coordinate ring of $H(r,n)$ is a finitely generated subalgebra of a polynomial ring, generated by the maximal minors of a generic Hankel matrix. For $r = 1$, the equations of $H(1,n)$ can be obtained by results in [1]. In [3], a combinatorial approach, in order to establish the equations of $H(r,n)$, is given for $r = 2$. Nevertheless,
the author is able to do only partial liftings, that for \( n = 5 \), become relations. The arguments are the Groebner bases theory, the Sagbi bases theory and the binomial relations given by Machado in [7], for any \( n \) and \( r \). The aim of this paper is to study the normality of the affine semigroup under the initial algebra of the coordinate ring of \( H(r, n) \), that we call toric deformation of \( H(r, n) \).

It is proved in [1] that the coordinate ring of \( H(1, n) \) and its deformation are normal. In this paper we study the deformation of \( H(r, n) \), by combinatorial arguments, already applied to \( G(r, n) \) in [8]. After some definitions and known results given in N. 1, we enounce in N. 2 a criterium established by B. Sturmfels in [8] and we apply it to equations of the deformation given by Machado. Hence we deduce the normality of the semigroup under the deformation and other related properties.

**Preliminaries and main results**

In this section we recall some results that we shall utilise in order to establish the normality of the deformation of \( H(r, n) \), for \( r = 1, 2 \). In the following, notations and results from the book of B. Sturmfels ([8], Chap. 11, 13) will be used.

We call \( A(r, n) \) the configuration of lattice vectors given by the exponents of the monomials generating the Sagbi basis of \( G(r, n) \) and let \( A \) be the affine semigroup generated by \( A(r, n) \), said the semigroup of the deformation of \( G(r, n) \).

We call \( B(r, n) \) the configuration of lattice vectors given by the exponents of the monomials generating the initial monomials of the polynomial generators of the coordinate ring of \( H(r, n) \) (not necessarily a Sagbi basis for this ring) and let \( B \) be the affine semigroup generated by \( B(r, n) \). Being \( H(r, n) \) a subvariety of \( G(r, n) \), then \( B(r, n) \) is a subconfiguration of \( A(r, n) \), \( B \) a subsemigroup of \( A \) and \( B \) is said the semigroup of the deformation of \( H(r, n) \).

The semigroup \( A \) was studied intensively and we can find results in [8]. The aim of this paper is to study properties of the semigroup \( B \) of the toric deformation \( H(r, n) \), for \( r = 1, 2 \). Concerning the normality, we will try to apply the following:

**Theorem 1.1** (Criterium 1, [8], Chap. 13). Let \( T \) be a finite subset of \( \mathbb{Z}^d \). Let \( I_T \) be a toric ideal (generated by binomials) and let \( K[T] = K[T_1, \ldots, T_n]/I_T \) be the correspondent toric ring. If there exists a term order \( < \) on the monomials of \( K[T_1, \ldots, T_n] \) such that \( \text{in}_<(I_T) \) is generated by square free monomials, then the toric variety \( X_T \) defined by the set \( T \) is normal.

**Proposition 1.2.** Let \( T \) be a finite subset of \( \mathbb{Z}^d \). Then the following facts are equivalent:
1. the affine toric variety $X_T$ is normal;

2. the integral domain $K[T] = K[T_1,\ldots,T_n]/I_T$ is integrally closed (in its fractions field);

3. the semigroup $NT$ generated by $T$ is normal.

Proof. See [8], Proposition 13.5.

From the computational point of view, we recall:

**Theorem 1.3.** Let $I_T$ be the toric ideal (generated by binomials) of the toric ring $K[F] = K[T_1,\ldots,T_n]/I_T$, being $F$ the set of monomials with support is $T$, and let $I$ be the ideal of a polynomial ring on $K$, whose $I_T$ is the deformation. Suppose $F$ is a Sagbi basis. Then we have the following facts:

1. Every reduced Groebner basis $G$ of $I_T$ lifts to a reduced Groebner basis $G$ of $I$;

2. every regular triangulation of $T$ is an initial complex of the ideal $I$;

3. the state polytope of $I_T$ is a face of the state polytope of the homogenization of $I$.

Proof. See [8], Corollary 11.6.

For the Grassmannian variety $G(r,n)$ all facts hold, in the sense that we have:

**Theorem 1.4.** Let $G(r,n)$ be the projective variety consisting of $r$–planes of $\mathbb{P}^n$. Then we have the following facts:

1. there exists a toric deformation $G^{(1,n)}$ of the variety $G(1,n)$ into the projective toric variety defined by the configuration $A(1,n)$;

2. the initial ideal of $I_{A(r,n)}$ is square-free;

3. the toric variety defined by $I_{A(r,n)}$ is projectively normal.

Proof. We can find in a vast literature, we refer to [8], Chap. 11.

For the variety $H^{(1,n)}$, deformation of $H(1,n)$, the normality is proved in [1]. Moreover, we point out from previous results the following geometric implications:

**Theorem 1.5.** Let $H(1,n)$ be the projective variety consisting of Hankel lines of $\mathbb{P}^n$. Then we have:
1. there exists a toric deformation $H^{(1,n)}$ of the Hankel variety $H(1,n)$ into the projective toric variety defined by the configuration $B(1,n)$;

2. the initial ideal of $I_{B(r,n)}$ is square-free;

3. the toric variety defined by $I_{B(r,n)}$ is projectively normal.

Proof. 1. The toric deformation can be obtained by employing the weight vectors ([4], [8]).

2. Since the adopted term order selects in each binomial generator of $T_{B(r,n)}$ as initial term the product of the indeterminates that are in the main diagonal of a generic Hankel matrix, that monomial is squarefree.

3. The assertion follows by Theorem 1.1 and Proposition 1.2, being $I_T$ homogeneous.

Concerning the study of the deformation of the Hankel variety $H(r,n)$, for $r > 1$ and any $n$, we don’t have the relations and therefore we can’t verify directly the normality of the deformation $H^{(r,n)}$. For $r = 2$ and $n = 5$, we have all the relations [3] and we can proceed by application of Theorem 1.2. In the following, we shall assume to be true the following:

**Hankel Conjecture (HC):** There exists a toric deformation $H^{(r,n)}$ of the Hankel variety $H(r,n)$ of the $r$–Hankel planes of $\mathbb{P}^n$ into the projective toric variety defined by the configuration $A(r,n)$, having the toric ideal $I_{A(r,n)}$ generated by the binomial relations given by Machado in [7].

**Theorem 2.1.** Let $H(r,n)$ be a Hankel variety of $\mathbb{P}^n$ that verifies HC. Then the following facts hold:

1. the initial ideal of $I_{B(r,n)}$ is square-free;

2. the toric variety defined by the ideal $I_{B(r,n)}$ is projectively normal.

Proof. 1. All binomials that define the toric deformation given by Machado in [7] are a Groebner basis for $I_{B(r,n)}$ ([7]) and the initial term of each binomial relation, with respect a term order that selects the product of the indeterminates of the main diagonal of a maximal minor in a generic Hankel matrix, is square-free, then the assertion follows, by Theorem 1.1 and Proposition 1.2, being $I_{B(r,n)}$ homogeneous.

We conclude with some open problems: For the Hankel variety analogue problems concerning to the Grassmann variety ([8], Chap. 11) arise, already for $H(1,n)$, as:
1. Does the Hankel-Plucker ideal $I(1,n)$ have initial ideals that are not squarefree?

2. What is the maximum degree appearing in any reduced Groebner basis for the Hankel-Plucker ideal $I(1,n)$?

Same problems arise for $H(2,n)$.

References


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