Asymptotic Theory for Vibrations of Composite Plates

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Abstract

A new theory for harmonic oscillations of thin composite plates is suggested. A main idea of the theory is to derive a closed equation system for composite plates from the general equations for steady oscillations of elastic solids in the three-dimensional statement with the asymptotic homogenization method. Any hypotheses on stress and displacement distributions through a plate thickness are not used. Unlike the classical method, our method is applied to thin multilayered plates without any periodicity. Recurrent chains of local problems of vibrations were deduced with the help of the asymptotic homogenization method, and closed-form solutions of the problems were found for thin composite plates. The method allows us to compute all six stresses’ distributions in a plate including normal through-thickness and shear interlayer stresses. An example of solving the problem on free flexural oscillations of a composite plate is shown, an asymptotic solution of the problem was found. Computations by the developed method and by finite-element solving the three-dimensional problem on free vibrations were compared. The developed theory for harmonic oscillations of composite plates allows us to calculate the natural frequencies and all six stresses in the plate up to a high accuracy.

Keywords: thin composite plate, asymptotic homogenization method, harmonic vibrations, finite element method
Introduction

Thin composite plates are widely used in different elements of aircraft, rocket and marine structures, which are subjected to the action of different types of vibrations. One of peculiarities of composite multilayered plates is the presence of low elastic characteristics of interlayer shear and transverse tension in comparison with longitudinal elastic moduli in a layers’ plane. The stresses may affect the forms of natural frequencies of plate oscillations and the energy dissipation in vibrations. In addition, under certain vibration levels there can appears a destruction of composite plates; it occurs along interlayer and transverse directions. For thin plates, the classical asymptotic homogenization method cannot be applied, because a 3D-periodic structure cannot be introduced for such plates. Some existing modifications [5] are based on introducing the hypotheses on a distribution character of displacements through a plate thickness, similarly to the Kirchhoff–Love and Mindlin–Timoshenko plate theories. Works [3, 4, 6] develop the asymptotic theory of multilayered plates, which does not contain any assumptions on displacement and stress distributions through a plate thickness.

Preliminaries and Main Results

Main assumptions of the asymptotic theory for thin plates. Consider a multilayered plate with a constant thickness and introduce a small parameter $\varkappa = h/L \ll 1$ (the ratio of a plate thickness $h$ to a characteristic plate size $L$, for example, to its maximum length). In addition, introduce global $x_k$ and local $\xi$ coordinates: $x_k = \tilde{x}_k/L$, $\xi = x_3/\varkappa (k = 1, 2, 3)$, where $\tilde{x}_k$ are usual Descartes coordinates oriented in such a way that the axis $O\tilde{x}_3$ is directed along the normal to the external and internal plate planes, and the axes $O\tilde{x}_1$ and $O\tilde{x}_2$ belong to the middle surface of the plate. We assume that there exist two scales of changing displacements $u_k$: the first of them is calculated along directions $O\tilde{x}_1$ and $O\tilde{x}_2$, and the second one — along the direction $O\tilde{x}_3$. For the plate, consider the following three-dimensional linear elasticity problem under steady vibrations [2]:

$$\nabla_j \sigma_{ij} + \rho \omega^2 u_i = 0, \quad \varepsilon_{ij} = (1/2)(\nabla_j u_i + \nabla_i u_j), \quad \sigma_{ij} = C_{ijkl} \varepsilon_{kl}, \quad (1)$$

$$\Sigma_{\pm} : \sigma_{i3} = -\varkappa^3 \rho_\pm \delta_{i3}; \quad \Sigma_T : \ u_i = u_{ei}; \quad \Sigma_S : [\sigma_{i3}] = 0, [u_3] = 0,$$
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where \( \sigma_{ij} \) are stress tensor components, \( \varepsilon_{ij} \) are strain tensor components, \( u_j \) are displacement vector components, \( \nabla_j = \partial / \partial \tilde{x}_j \) is the differentiation operator with respect to Descartes coordinates, \( \omega \) is the frequency of forced vibrations, \( C_{ijkl}(\xi) \) are elastic moduli tensor components, \( \rho(\xi) \) is density of the plate layers.

Let us make the key assumption that the pressure \( \tilde{p}_\pm \) on external and internal plate surfaces is of the smallness order \( O(\varkappa^2) \) (i.e. \( \tilde{p}_\pm = \varkappa^2 p_\pm \)).

The problem (2) contains a local coordinate \( \xi \) and a small parameter \( \varkappa \) in its boundary conditions, thus, its solution is sought as the asymptotic expansions

\[
u_k = u_k^0(x_I) + \varkappa u_k^{(1)}(x_I, \xi) + \varkappa^2 u_k^{(2)}(x_I, \xi) + \varkappa^3 u_k^{(3)}(x_I, \xi) + \ldots
\]

Indices \( I, J, K, L \) take values 1 and 2; and \( i, j, k, l - 1, 2 \) and 3.

Substituting (2) into the Cauchy relations in (1), we find

\[
\varepsilon_{ij} = \varepsilon_{ij}^{(0)} + \varkappa \varepsilon_{ij}^{(1)} + \varkappa^2 \varepsilon_{ij}^{(2)} + \ldots,
\]

where

\[
\varepsilon_{ij}^{(0)} = (1/2)(u_{i,j}^0 + u_{j,i}^0), \quad \varepsilon_{ij}^{(0)} = (1/2)(u_{33}^0 + u_{i,j}^0), \quad \varepsilon_{33}^{(0)} = u_{33}^0, \quad \varepsilon_{ij}^{(1)} = (1/2)(u_{i,j}^{(1)} + u_{j,i}^{(1)}), \quad \varepsilon_{33}^{(1)} = u_{33}^{(1)}, \quad \varepsilon_{ij}^{(2)} = (1/2)(u_{i,j}^{(2)} + u_{j,i}^{(2)}), \quad \varepsilon_{33}^{(2)} = u_{33}^{(2)}, \quad \text{etc.}
\]

Substituting (3) into the Hooke law in (1), we obtain

\[
\sigma_{ij} = \sigma_{ij}^{(0)} + \varkappa \sigma_{ij}^{(1)} + \varkappa^2 \sigma_{ij}^{(2)} + \ldots.
\]

where

\[
\sigma_{ij}^{(0)} = C_{IJKL} \varepsilon_{KL}^{(0)} + C_{IJK33} \varepsilon_{k3}^{(0)}, \quad \sigma_{i3}^{(0)} = C_{3IKL} \varepsilon_{KL}^{(0)} + C_{i3k3} \varepsilon_{k3}^{(0)}, \quad \text{etc.}
\]

Local problems. Substituting (2), (3) and (4) into (1), we obtain

\[
(1/\varkappa)\sigma_{i33}^{(0)} + (\sigma_{i,j}^{(0)} + \sigma_{i3}^{(1)} + \rho \omega^2 u_{i}^{(0)}) + \varkappa(\sigma_{i,j}^{(1)} + \sigma_{i3}^{(2)} + \rho \omega^2 u_{i}^{(1)}) + \varkappa^2(\sigma_{i,j}^{(2)} + \sigma_{i3}^{(3)} + \rho \omega^2 u_{i}^{(2)}) + \ldots = 0,
\]

\[
\Sigma_{3\pm} : \quad \sigma_{i3}^{(0)} + \varkappa \sigma_{i3}^{(1)} + \ldots = -\varkappa^2 p_\pm \delta_{i3}; \quad \Sigma_T : \quad u_i = u_i^{(0)} + \varkappa u_i^{(1)} + \ldots = u_{ei}.
\]

Putting terms with \( \varkappa^{-1} \) equal to zero and with the remaining powers of \( \varkappa \) equal to some values \( h_i^{(0)} \), \( h_i^{(1)} \) and \( h_i^{(2)} \) independent of \( \xi \), we get the following recurrent sequence of local problems. The zero-level problem has the form

\[
\sigma_{i3}^{(0)} = 0, \quad \sigma_{i3}^{(0)} = C_{3IKL} \varepsilon_{KL}^{(0)} + C_{i3k3} \varepsilon_{k3}^{(0)}, \quad \varepsilon_{i,j}^{(0)} = (1/2)(u_{i,j}^{(0)} + u_{j,i}^{(0)}), \quad \varepsilon_{i3}^{(0)} = (1/2)(u_{33}^{(0)} + u_{i,j}^{(0)}), \quad \varepsilon_{33}^{(0)} = u_{33}^{(0)},
\]

\[
\Sigma_{3\pm} : \quad \sigma_{i3}^{(0)} = 0; \quad \Sigma_S : \quad [\sigma_{i3}^{(0)}] = 0, [u_i^{(0)}] = 0, [u_i^{(1)}] = 0;
\]

the first-level problem:

\[
\sigma_{i3}^{(1)} + \rho \omega^2 u_{i}^{(0)} = h_i^{(0)}, \quad \sigma_{i3}^{(1)} = C_{3IKL} \varepsilon_{KL}^{(1)} + C_{i3k3} \varepsilon_{k3}^{(1)}, \quad \varepsilon_{i,j}^{(1)} = (1/2)(u_{i,j}^{(1)} + u_{j,i}^{(1)}), \quad \varepsilon_{i3}^{(1)} = (1/2)(u_{33}^{(1)} + u_{i,j}^{(1)}), \quad \varepsilon_{33}^{(1)} = u_{33}^{(1)},
\]

\[
\Sigma_{3\pm} : \quad \sigma_{i3}^{(1)} + \rho \omega^2 u_{i}^{(1)} = h_i^{(1)}, \quad \Sigma_S : \quad [\sigma_{i3}^{(1)}] = 0, [u_i^{(0)}] = 0, [u_i^{(1)}] = 0;
\]

the second-level problem:
\[ \Sigma_{3\pm} : \sigma_{i3}^{(1)} = 0; \quad \Sigma_S : [\sigma_{i3}^{(1)}] = 0, \quad [u_i^{(2)}] = 0, \quad \langle u_i^{(2)} \rangle = 0; \]

the second-level problem:

\[ \sigma_{ij}^{(2)} + \sigma_{ij,j}^{(1)} + \rho \omega^2 u_i^{(1)} = h_i^{(1)}, \quad \sigma_{i3}^{(2)} = C_{i3kL} \varepsilon_{KL}^{(2)} + C_{i3k3} \varepsilon_{k3}^{(2)}, \]

\[ \varepsilon_{ij}^{(2)} = (1/2)(u_i^{(2)} + u_j^{(2)}), \quad \varepsilon_{ij}^{(1)} = (1/2)(u_i^{(2)} + u_{i/3}^{(3)}), \quad \varepsilon_{33}^{(2)} = u_{3/3}^{(3)}, \quad (9) \]

\[ \Sigma_{3\pm} : \sigma_{i3}^{(2)} = 0; \quad \Sigma_S : [\sigma_{i3}^{(2)}] = 0, \quad [u_i^{(3)}] = 0, \quad \langle u_i^{(3)} \rangle = 0; \]

the third-level problem:

\[ \sigma_{i3}^{(3)} + \sigma_{i3,j}^{(1)} + \rho \omega^2 u_i^{(2)} = h_i^{(2)}, \quad \sigma_{i3}^{(3)} = C_{i3kL} \varepsilon_{KL}^{(3)} + C_{i3k3} \varepsilon_{k3}^{(3)}, \]

\[ \varepsilon_{ij}^{(3)} = (1/2)(u_i^{(2)} + u_j^{(2)}), \quad \varepsilon_{ij}^{(3)} = (1/2)(u_i^{(3)} + u_{i/3}^{(3)}), \quad \varepsilon_{33}^{(3)} = u_{3/3}^{(3)}, \quad (10) \]

\[ \Sigma_{3\pm} : \sigma_{i3}^{(3)} = p_{\pm} \delta_{i3}; \quad \Sigma_S : [\sigma_{i3}^{(3)}] = 0, \quad [u_i^{(3)}] = 0, \quad \langle u_i^{(3)} \rangle = 0; \]

e tc., where \( \langle u_i^{(1)} \rangle = \int_{0.5}^{0.5} u_i^{(3)} d\xi \). Then equations (8) take the form

\[ h_i^{(0)} + \chi h_i^{(1)} + \chi^2 h_i^{(2)} + \ldots = 0. \quad (11) \]

**Solving the zero-level problem.** Since problems (7)–(9) are one-dimensional in local variable \( \xi \), their solution can be found in the analytic way. A solution of equilibrium equations in (7) has the form

\[ \sigma_{i3}^{(0)} = 0 \quad \forall \xi : -0.5 < \xi < 0.5. \quad (12) \]

Substituting (5) for \( \sigma_{i3}^{(0)} \) into (12), we get

\[ C_{i3kL} \varepsilon_{KL}^{(0)} + C_{i3k3} \varepsilon_{k3}^{(0)} = 0, \]

where \( C_{i3k3} \) is the matrix inverse to \( C_{i3k3} \). Integrating with conditions \( \langle u_i^{(1)} \rangle = 0 \), we find

\[ u_i^{(1)} = -\xi u_i^{(0)} + 2\xi \varepsilon_{KL}^{(0)} \left( \int_{-0.5}^{\xi} C_{i3k3}^{-1} C_{i3kL} d\xi \right) - \int_{-0.5}^{\xi} C_{i3k3}^{-1} C_{i3kL} d\xi, \quad (13) \]

\[ u_i^{(1)} = \varepsilon_{KL}^{(0)} \left( \int_{-0.5}^{\xi} C_{i3k3}^{-1} C_{i3kL} d\xi \right) - \int_{-0.5}^{\xi} C_{i3k3}^{-1} C_{i3kL} d\xi. \quad (14) \]

Substituting (13) into (5), we get that unlike \( \sigma_{i3}^{(0)} \), stresses \( \sigma_{i3}^{(1)} \) are nonzero:

\[ \sigma_{ij}^{(0)} = C_{i1jkL} \varepsilon_{KL}^{(0)}, \quad \sigma_{ij}^{(1)} = C_{i1jkL} - C_{i1jk3} C_{33k3}^{-1} C_{33kL}. \quad (15) \]

**Solutions of the first-, second- and third-level problems.** The steady-vibration equations (8), (9) and (10) with boundary conditions at \( \Sigma_S \) and \( \xi = -0.5 \) have the solutions

\[ \sigma_{i3}^{(1)} = -\int_{-0.5}^{\xi} (\sigma_{ij}^{(1),j} + \rho \omega^2 u_i^{(0)}) d\xi + h_i^{(0)} (\xi + 0.5), \quad (16) \]

\[ \sigma_{i3}^{(2)} = -\int_{-0.5}^{\xi} (\sigma_{ij}^{(1),j} + \rho \omega^2 u_i^{(1)}) d\xi + h_i^{(1)} (\xi + 0.5), \quad (17) \]

\[ \sigma_{i3}^{(3)} = -p_{\pm} \delta_{i3} - \int_{-0.5}^{\xi} (\sigma_{ij}^{(2),j} + \rho \omega^2 u_i^{(2)}) d\xi + h_i^{(2)} (\xi + 0.5). \quad (18) \]

The solution existence conditions (16)–(18) of problems (8)–(10) lead to the system
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Due to (19)–(20), stresses $\sigma_{i3}^{(m)}$ (16)–(18) take the form

$$
\sigma_{i3}^{(1)} = \int_{-0.5}^{\xi} \left( (\sigma_{i,j,j}^{(0)}) - \sigma_{i,j,j}^{(0)} + (\rho - \rho)\omega^2 u_i^{(0)} \right) d\xi,
$$

$$
\sigma_{i3}^{(2)} = \int_{-0.5}^{\xi} \left( (\sigma_{i,j,j}^{(0)}) - \sigma_{i,j,j}^{(0)} + \omega^2(\rho u_i^{(1)} - \rho u_i^{(1)}) \right) d\xi,
$$

$$
\sigma_{i3}^{(3)} = -(p_+ + \Delta p(\xi + 0.5))\delta_{ij} + \int_{-0.5}^{\xi} \left( (\sigma_{i,j,j}^{(2)}) - \sigma_{i,j,j}^{(2)} + \omega^2(\rho u_i^{(2)} - \rho u_i^{(2)}) \right) d\xi.
$$

Substituting (15) into (21), we get

$$
\sigma_{i3}^{(1)} = \varepsilon_{KL,j}^{(0)} \int_{-0.5}^{\xi} (C_{i,j,k}^{(0)} - C_{i,j,k}^{(0)}) d\xi + \int_{-0.5}^{\xi} ((\rho - \rho)\omega^2 u_i^{(0)} d\xi,
$$

$$
\varepsilon_{i,j}^{(3)} = \int_{-0.5}^{\xi} ((\rho - \rho)\omega^2 u_i^{(0)} d\xi.
$$

Unlike the quasistatic problem [2], for steady vibrations the stress $\sigma_{i3}^{(1)}$ is nonzero. From the fourth group of (5) we can express the strains

$$
\varepsilon_{k3}^{(1)} = -C_{k33}^{-1} C_{i3KL}^{(1)} + \varepsilon_{KL,j}^{(0)} C_{k33}^{-1} \int_{-0.5}^{\xi} (C_{i,j,k}^{(0)} - C_{i,j,k}^{(0)}) d\xi + \omega^2 C_{k33}^{-1} u_i^{(0)} \int_{-0.5}^{\xi} ((\rho - \rho)\omega^2 u_i^{(0)} d\xi.
$$

Substituting (24) into the third group of (5), we find the remaining first-level stresses

$$
\sigma_{i,j}^{(1)} = C_{i,j,k}^{(0)} \varepsilon_{KL,j}^{(0)} + N_{i,j,k}^{(0)} \varepsilon_{KL,j}^{(0)} + \omega^2 G_{i,j,k}^{(0)} u_i^{(0)},
$$

$$
N_{i,j,k}^{(0)} = C_{i,j,k}^{(0)} C_{k33}^{-1} \int_{-0.5}^{\xi} (C_{p,m,k}^{(0)} - C_{p,m,k}^{(0)}) d\xi,
$$

$$
G_{i,j,k}^{(0)} = C_{i,j,k}^{(0)} C_{k33}^{-1} \int_{-0.5}^{\xi} ((\rho - \rho)\omega^2 u_i^{(0)} d\xi.
$$

With account of (8) and (14),

$$
\varepsilon_{KL,j}^{(1)} = \varepsilon_{KL,j}^{(0)} + \Phi_{KL,MN,S}\varepsilon_{MN,S}^{(0)},
$$

$$
\eta_{KL} = -u_{KL,j}^{(0)}, \quad \Phi_{KL,MN,S}^{(0)} = \tilde{\Phi}_{KL,MN,S}^{(0)} - (\tilde{\Phi}_{KL,MN,S}^{(0)}),
$$

$$
\tilde{\Phi}_{KL,MN,S}^{(0)} = -\int_{-0.5}^{\xi} (C_{K33}^{-1} \delta_{SL} + C_{L33}^{-1} \delta_{SK}) C_{33MN} d\xi.
$$

Due to (26), expressions (25) yield

$$
\sigma_{i,j}^{(1)} = \xi C_{i,j,k}^{(0)} \eta_{KL} + \tilde{N}_{i,j,k}^{(0)} \varepsilon_{KL,j}^{(0)} + \omega^2 G_{i,j,k}^{(0)} u_i^{(0)}.
$$

The second-level displacement $u_i^{(2)}$ can be determined by the third formula of (5):
After integrating this expression with account of \( u_{i/3}^{(2)} = -u_{3,i}^{(1)} + 2C_{133}^{-1}(\sigma_{i3}^{(1)} - C_{i3KL}\varepsilon_{KL}^{(1)}) \).

Substituting (19) into (11), we have

\[
\langle\sigma_{iJ,J}\rangle + \langle\rho\rangle\omega^{2}\langle u_{i}^{(0)}\rangle + \chi\langle\sigma_{iJ,i}^{(1)} + \omega^{2}\langle pu_{i}^{(1)}\rangle\rangle + \chi^{2}\langle\sigma_{iJ,i}^{(1)} + \omega^{2}\langle pu_{i}^{(2)}\rangle - \Delta p\delta_{i3}\rangle + \ldots = 0.
\]

Multiplying (6) by \( \chi\varepsilon\sigma_{iJ,i}^{(0)} + \omega^{2}\langle pu_{i}^{(1)}\rangle\) and integrating them over the thickness, we get

\[
\chi\langle\xi\sigma_{iJ,i}^{(0)} + \omega^{2}\langle pu_{i}^{(1)}\rangle\rangle - \langle\xi\sigma_{iJ,i}^{(1)}\rangle + \chi^{2}\langle\xi\sigma_{iJ,i}^{(1)} + \omega^{2}\langle pu_{i}^{(2)}\rangle - \langle\xi\sigma_{iJ,i}^{(2)}\rangle\rangle + \ldots = 0.
\]

Introduce the notation

\[
T_{IJ} = \langle\sigma_{iJ}^{(0)}\rangle + \chi\langle\sigma_{iJ}^{(1)}\rangle + \ldots, \quad Q_I = \chi\langle\sigma_{iJ}^{(1)}\rangle + \chi^{2}\langle\sigma_{iJ}^{(2)}\rangle + \ldots, \quad M_{IJ} = \chi\langle\sigma_{iJ}^{(0)}\rangle + \ldots,
\]

\[
\bar{p}U_i = \langle\rho\rangle u_{i}^{(0)} \chi\varepsilon\sigma_{iJ,i}^{(0)} + \omega^{2}\langle pu_{i}^{(1)}\rangle + \chi^{2}\langle\sigma_{iJ,i}^{(2)}\rangle + \ldots, \quad \bar{p}\Gamma_I = \chi\langle\rho u_{i}^{(1)}\rangle + \chi^{2}\langle\rho u_{i}^{(2)}\rangle + \ldots.
\]

Retaining only the principal terms of the asymptotic expansions, we get

\[
U_i = u_{i}^{(0)}, \quad \bar{p}\Gamma_I = \chi\langle\rho u_{i}^{(1)}\rangle\xi = -Ru_{3,i}^{(0)} + \varepsilon_{KL}^{(0)}R_{IKL},
\]

\[
R_{IKL} = 2\chi\langle\int\xi C_{133}^{-1}C_{i3KL}d\xi\rangle + 2\chi^{2}\langle\rho\varepsilon\int\xi C_{133}^{-1}C_{i3KL}d\xi\rangle, \quad R = \chi\langle\rho\varepsilon^{2}\rangle.
\]

Thus, equations (30)–(31) can be written, as usual in the plate theory, in the form of equilibrium and moments’ equations under steady vibrations

\[
T_{I,J} + \rho\varepsilon^{2}U_I = 0, \quad Q_{I,J} + \rho\varepsilon^{2}U_3 = \Delta \bar{p}, \quad M_{I,J} - Q_I + \rho\varepsilon^{2}\Gamma_I = 0. \quad (33)
\]

These are the desired averaged equations for steady vibrations of a multilayer plate. Substituting (15), (23), and (29) into integrals of (32), we get

\[
T_{IJ} = \bar{C}_{IJKL}\varepsilon_{KL}^{(0)} + B_{IKL}\eta_{KL} + K_{IJKLM}\varepsilon_{KL,M}^{(0)} + \omega^{2}\bar{G}_{IJ,i}u_{i}^{(0)},
\]

\[
M_{IJ} = B_{IKL}\varepsilon_{KL}^{(0)} + D_{IKL}\eta_{KL} + K_{IJKLM}\varepsilon_{KL,M}^{(0)} + \omega^{2}\bar{G}_{IJ,i}u_{i}^{(0)},
\]

\[
Q_I = K_{IJKL}\varepsilon_{KL,J}^{(1)} + \omega^{2}\bar{G}_{IJ,i}u_{i}^{(0)} + \chi^{2}\langle\sigma_{iJ,i}^{(2)}\rangle.
\]

In particular, when the following conditions are satisfied: 1) layers are symmetric relative to the plane \( \xi = 0 \), and 2) thicknesses of all the layers are the same, functions of form \( \langle C_{PMKL}^{(0)}\rangle - C_{PMKL}^{(0)} \) are symmetric relative to the plane, and functions \( \int_{-0.5}^{0.5} (\langle C_{PMKL}^{(0)}\rangle - C_{PMKL}^{(0)}) \) are antisymmetric, thus, \( B_{IKL} = 0, \quad K_{IJKLM} = 0, \quad K_{IJKL} = 0, \quad \bar{G}_{I,J,i} = 0, \) and (34)–(36) take a simpler form

\[
T_{IJ} = \bar{C}_{IJKL}\varepsilon_{KL}^{(0)}, \quad M_{IJ} = D_{IKL}\eta_{KL} + \bar{K}_{IJKLM}\varepsilon_{KL,M}^{(0)} + \omega^{2}\bar{G}_{IJ,i}u_{i}^{(0)}.
\]

\( (37) \)
Equations (34)–(36) includes strains $\varepsilon_{KL}^{(0)}$, curvatures $\eta_{KL}$ and strain gradients $\varepsilon_{KL,N}^{(0)}$, depending on functions $u_1^{(0)}$, $u_3^{(0)}$ of global variables $x_1$, and $\varepsilon_{ij}^{(0)} = (1/2)(u_{i,j}^{(0)} + u_{j,i}^{(0)})$, $\eta_{KL} = -u_{3,KL}^{(0)}$. Substituting (34)–(36) into (33), we get

$$\begin{align*}
\tilde{C}_{IJKL}u_{I,J}^{(0)} - B_{IJKL}u_{3,KL}^{(0)} + K_{IJKL}u_{K,LMJ}^{(0)} + \omega^2(\tilde{G}_{IJI} + \langle \rho \rangle \delta_3)u_{i,J}^{(0)} &= 0, \\
B_{IJKL}u_{K,LJI}^{(0)} - D_{IJKL}u_{3,KL}^{(0)} + \tilde{K}_{IJKL}u_{K,LMJ}^{(0)} + \\
+ \omega^2\tilde{G}_{IJI}u_{i,J}^{(0)} - \omega^2(Ru_{3,JI}^{(0)} - u_{K,JI}^{(0)}R_{I,KL} + \tilde{p}u_3^{(0)}) &= \Delta \tilde{p}.
\end{align*}$$

(38)

After solving the equations (38) and finding $u_i^{(0)}$ and $u_3^{(0)}$, we can calculate strains and then stresses $\sigma_{I,J}^{(0)}$ by (15). It was derived that stresses $\sigma_{I3}^{(0)}$ and $\sigma_{33}^{(0)}$ in the plate are identical zero. Nonzero values of shear stresses appear for the next term of the asymptotic expansion, namely for $\sigma_{I3}^{(1)}$, due to (23). For the transverse stress, the first nonzero value in the asymptotic expansion is the value of $\sigma_{33}^{(1)}$ calculated by (23), and the next terms of expansion, i.e. $\sigma_{33}^{(2)}$ and $\sigma_{33}^{(3)}$, are determined by (22):

$$\begin{align*}
\sigma_{33} &= \kappa \int_{-0.5}^{\xi} ((\rho) - \rho)\omega^2u_3^{(0)}d\xi + \kappa_2 \int_{-0.5}^{\xi} ((\sigma_{3,j,j}^{(1)}) - \sigma_{3,j,j}^{(1)} + ((\rho) - \rho)\omega^2u_3^{(1)})d\xi + \\
+ \kappa_2^2 \left(-p_{e} - \Delta p(\xi + 0.5) + \int_{-0.5}^{\xi} ((\sigma_{3,j,j}^{(2)}) - \sigma_{3,j,j}^{(2)} + \omega^2((\rho u_3^{(2)} - \rho u_3^{(2)}))d\xi\right), \\
\sigma_{I3} &= \kappa \int_{-0.5}^{\xi} ((\sigma_{I,j,j}^{(0)}) - \sigma_{I,j,j}^{(0)} + ((\rho) - \rho)\omega^2u_3^{(0)})d\xi + \kappa_2 \int_{-0.5}^{\xi} ((\sigma_{I,j,j}^{(1)}) - \\
- \sigma_{I,j,j}^{(1)} + \omega^2((\rho u_3^{(1)} - \rho u_3^{(1)}))d\xi. 
\end{align*}$$

(39)

**Flexural vibrations of a symmetric multilayered composite plate.**

Consider the classical problem on steady vibrations of a rectangular plate under the uniformly distributed pressure. Let the plate layers be located symmetrically relative to the plane $\xi = 0$, then

$$u_I^{(0)} = 0, \quad \varepsilon_{KL}^{(0)} = 0, \quad T_{IJ} = 0, \quad \sigma_{I,I}^{(0)} = 0,$$

(40)

and nonzero unknown functions are only the functions $u_3^{(0)}(x)$, $M_{11}(x)$ and $Q_1(x)$ ($x = x_1$). Equations (33) and (35) take the form

$$Q_{1,1} + \tilde{\rho}\omega^2u_3^{(0)} = \Delta \tilde{p}, \quad M_{11,1} - Q_1 - \omega^2Ru_3^{(0)} = 0, \quad M_{11} = D_{1111}\eta_{11} + \omega^2\tilde{G}_{113}u_3^{(0)}.$$

Eliminating the crossing force in the first two equations, we obtain $M_{11,11} + \omega^2(\tilde{\rho}u_3^{(0)} - Ru_3^{(0)}) = \Delta \tilde{p}, \quad M_{11} = -D_{1111}u_3^{(0)} + \omega^2\tilde{G}_{113}u_3^{(0)}$. So we find the final differential equation for vibrations of a multilayer plate:

$$D_{1111}u_{3,1111}^{(0)} + \omega^2R(1 - \tilde{G}_{113}/R)u_{3,111}^{(0)} - \omega^2\tilde{p}u_3^{(0)} + \Delta \tilde{p} = 0.$$

(41)
Solve equations (41) with boundary conditions of hinged plate ends \( x = 0 \) and \( x = 1 \): \( u_3^{(0)} = 0, \upsilon_{3,11}^{(0)} = 0 \). Due to (12) and (40), stresses \( \sigma_{12}^{(0)} \) and \( \sigma_{32}^{(0)} \) are zero, and stresses of the first- and second-level approximations have the form

\[
\sigma_{12}^{(1)} = -\xi C_{11}^{(0)} u_{3,11}^{(0)} + \omega^2 G_{113} u_3^{(0)}, \quad \sigma_{13}^{(1)} = \sigma_{33}^{(2)} = 0, \quad \sigma_{33}^{(1)} = u_3^{(0)} \int_{-0.5}^{\xi} ((\rho) - \rho) \omega^2 d\xi,
\]

\[
\sigma_{33}^{(2)} = u_{3,11}^{(0)} \int_{-0.5}^{\xi} ((\xi C_{1111}^{(0)} - \xi C_{1111}^{(0)})) d\xi + \omega^2 u_{3,11}^{(0)} \int_{-0.5}^{\xi} ((G_{1133} - G_{1133})) d\xi -
\]

\[
- \omega^2 u_{3,11}^{(0)} \int_{-0.5}^{\xi} ((\rho \xi) - \rho \xi) d\xi, \quad \sigma_{33}^{(3)} = \int_{-0.5}^{\xi} ((\sigma_{33,j}^{(2)}) - \sigma_{33,j}^{(2)}) + \omega^2 ((\rho \sigma_{33}^{(2)} - \rho \sigma_{33}^{(2)})) d\xi.
\]

When only the principal terms are taken into account, we obtain

\[
\sigma_{11} = -\chi ((\xi C_{1111}^{(0)} u_{3,11}^{(0)} + \omega^2 G_{1133} u_3^{(0)})),
\]

\[
\sigma_{33} = -\chi^2 u_{3,11}^{(0)} \int_{-0.5}^{\xi} ((\xi C_{1111}^{(0)} - \xi C_{1111}^{(0)})) d\xi + \chi \omega^2 u_{3,11}^{(0)} \int_{-0.5}^{\xi} ((G_{1133}) - G_{1133})) d\xi -
\]

\[
- \chi \omega^2 u_{3,11}^{(0)} \int_{-0.5}^{\xi} ((\rho \xi) - \rho \xi) d\xi, \quad \sigma_{33} = \int_{-0.5}^{\xi} ((\rho) - \rho) \omega^2 u_3^{(0)} d\xi -
\]

\[
- \chi^3 (p_+ + \Delta p (\xi + 0.5)) - \int_{-0.5}^{\xi} ((\sigma_{31,1}^{(2)}) - \sigma_{31,1}^{(2)}) d\xi - \omega^2 \int_{-0.5}^{\xi} ((\rho \sigma_{33}^{(2)} - \rho u_{3,11}^{(0)})) d\xi,
\]

\[
\sigma_{31,1} = -u_{3,11}^{(0)} \int_{-0.5}^{\xi} ((\sigma_{31}^{(2)}) - \sigma_{31}^{(2)}) d\xi + \omega u_{3,11}^{(0)} \int_{-0.5}^{\xi} ((G_{1131}) - G_{1131})) d\xi -
\]

\[
- \int_{-0.5}^{\xi} ((G_{1131}) - G_{1131})) d\xi - \omega^2 u_{3,11}^{(0)} \int_{-0.5}^{\xi} ((\xi \rho) - \xi \rho) d\xi -
\]

\[
- \int_{-0.5}^{\xi} ((\xi \rho) - \xi \rho) d\xi, \quad \sigma_{32}^{(2)} = \int_{-0.5}^{\xi} ((\xi C_{1111}^{(0)} - \xi C_{1111}^{(0)})) d\xi.
\]

**Computation by the theory and ANSYS.** To estimate an accuracy of the asymptotic theory (AT), we compared calculations by (42) and by the three-dimensional elasticity theory. For numerical solving we used the finite-element ANSYS with the 10-nodal finite element SOLID187. Figures 1 and 2 compare distributions of stresses given by the AT- and ANSYS-solutions for two different crossections: \( x = x_1 = [0.25; 0.5] \). The transverse dimensionless coordinate varies within the interval \([-0.5, 0.5]\): value \( \xi = 0.5 \) corresponds to the upper plane; values \( \xi = \pm 0.25 \) — the planes of layers’ contact. Since the layers’ materials were chosen to be orthotropic, two tangential stresses are absent in all the layers: \( \sigma_{12} = \sigma_{23} = 0 \). Distributions of the remaining four stresses \( \sigma_{13}, \sigma_{11}, \sigma_{22} \) and \( \sigma_{33} \), corresponding to the lower frequency of free oscillations and calculated by (42) and by ANSYS for the grid \( N = 12 \), are in sufficient agreement. The solution in Fig.1,b indicates that the zero tangential stress distribution \( \delta(\sigma_{13}) \) at \( x_1 = 0.5 \) in the FE-simulation proved to be close
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Fig.1. Distribution of $\sigma_{13}$ calculated by AT and ANSYS: $x_1 = 0.25$ (a); $x_1 = 0.5$ (b)

Fig.2. Stresses $\sigma_{11}$ (a) and $\sigma_{33}$ (b): $x_1 = 0.25$ (1); $x_1 = 0.5$ (2)

to zero computed; and maximum deviations from zero are smaller by three orders than the maximum value of the tangential stresses at $x_1 = 0.25$.

Conclusions

The theory for free oscillations of thin multilayered plates was developed with the asymptotic homogenization method applied to the general equations of oscillations of elastic bodies in the three-dimensional statement without any hypotheses on displacement and stress distributions through a plate thickness. Recurrent chains of local problems were formulated, and their solutions were obtained in the explicit form. The averaged problem of the plate vibration theory proves to be similar to the Kirchhoff–Love theory and differs from the last one only by the presence of the third-order derivatives of longitudinal displacements. Terms with these derivatives are nonzero only for plates with nonsymmetric location of layers through a plate thickness. The method allows us to calculate all six components of the stress tensor.
References


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