Mathematical Modeling of Longitudinal Oscillations Tapered Narrow Channel Wall under Pulsating Pressure of Highly Viscous Liquid

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Abstract

The longitudinal oscillations of tapered narrow channel wall under pulsating pressure of highly viscous incompressible liquid inside is set up and analytically solved. The problem in a flat setting for the regime of a stationary pulsating liquid movement in the tapered narrow channel with elastically fixed wall under the suggested harmonic pulsating pressure at its left edge is considered. The formulated boundary problem represents Navier-Stokes equations system for highly viscous incompressible liquid layer in narrow gap and the dynamic equation of elastic fixed channel wall. The conditions of liquid adhesion to impenetrable channel walls and the condition of free leakage of liquid at channel edges are presented in the paper as the boundary ones. The dimensionless variables of the problem under consideration and its small parameter are suggested as well. The relative mean thickness of liquid layer is taken as small parameter. The setting and solving hydroelasticity problem of tapered narrow channel with elastically fixed wall made it possible to find analytical expressions of channel wall displacements and pressure in highly viscous incompressible liquid. The amplitude frequency characteristic and phase frequency characteristic of elastically fixed tapered narrow channel wall are constructed as well.

Keywords: mathematical modeling, hydroelasticity, highly viscous liquid, tapered narrow channel, pressure pulsating, elastically fixed wall, oscillations
Introduction

The mathematical modeling problems of hydroelastic beams and plates oscillations are of theoretical and practical interest for modern mechanics and applied mathematics. For instance, reference [1] presents the study of oscillations of a plate plunged in an ideal incompressible liquid with a three surface. The bending oscillations of channel wall as a beam interacting with an ideal liquid, filling the channel are solved in [2]. The chaotic oscillations of a plate under its interaction with ideal incompressible flow are investigated in [3]. The vibration of a round plate on a free surface of an ideal incompressible liquid are investigated in [4]. The problem for case of the liquid section limited by a rigid bottom and cylindrical surface is considered. However, these studies do not allow damping in the considered oscillating systems because of the liquid viscosity neglected.

The dynamics of viscous incompressible liquid layer in the channel with absolutely solid walls is investigated in the frames of hydrodynamic lubrication theory [5, 6]. The oscillations of elastically fixed channel wall under pulsating viscous liquid layer inside are study in [7]. The hydroelastic oscillations of cantilevered beam in an unlimited volume of viscous incompressible liquid are investigated in [8]. The damping harmonically vibrating infinitely long beam on the viscous incompressible and compressible liquid are studied in [9]. The interaction of vibrating discs with viscous incompressible liquid layer between them is investigated in [10]. The analogous problem for the two vibrating plates is solved in [11]. Reference [12] carried out the study of hydroelastic beam oscillations in a viscous flow in application to piezoelectric element, which may be used for receiving energy from the flow. In reference [13] solved the problem of bending hydroelastic oscillations of a plate, forming the wall of a narrow channel with a pulsating layer of viscous incompressible liquid with an assigned harmonic law of pressure pulsating at one channel edge. The interaction of beam with viscous incompressible liquid layer on the basis of the one-mass system and the method of equivalent mass is investigated in [14]. The problem of dynamic interaction of the plate on the elastic foundation with pulsating viscous incompressible liquid layer is analytically solved in [15]. The longitudinal oscillations of plate in a viscous flow inside a channel formed by two parallel solid walls are studied in [16].

The objective of this paper is to study the effect of tapering the narrow channel with elastically fixed solid wall and to investigate the longitudinal hydroelastic wall oscillations under pulsating highly viscous liquid in such channels.

Mathematical formulation

Let us consider the channel in fig.1. It is formed by two plane walls 1 and 2 with a pulsating layer of highly viscous incompressible liquid 3, moving at the expense of the suggested pulsating pressure at the left edge. One of the side walls is oriented horizontally (wall 1) and the opposite wall 2 is tilted at the some angle.
Mathematical modeling of longitudinal oscillations...

with respect to the horizontal position. That fact can be described by parameter $\theta$. The wall $I$ is connected elastically and can move in a longitudinal direction. The channel wall $I$ has geometric sizes $b \times 2\ell$. Later, let us think, that channel size is $b \gg 2\ell$. The mean liquid layer thickness (the mean distance between the walls) represents $\delta_0$ and is considerably smaller than channel length $2\ell$. As a result of the pressure pulsating, there emerge the wall $I$ longitudinal oscillations. The leakage at channel edges can be considered as a spurt one in the cavity, filled with the same liquid. To be definite, let us consider, that the pressure in the left cavity is $p^*(\omega t)$, and in the right one is zero.

![Fig. 1. The scheme of tapered channel with an elastic fixed wall, which can be oscillate longitudinally](image)

The law of pressure $p^*(\omega t)$ at the left edge is presented as:

$$p^*(\omega t) = p_m f_p(\omega t), \quad f_p(\omega t) = \sin(\omega t),$$

where $p_m$ is the amplitude of pressure pulsating at the left edge; $\omega$ is the pulsating frequency; $f_p(\omega t)$ is the pressure change law.

Let us locate Cartesian coordinate in the center of wall $I$ (point O in the fig. 1). Taking into consideration, that $b \gg 2\ell$, we will further solve the problem of elastically fixed channel wall hydroelastic oscillations in a flat setting. The analyzed mechanical system includes strong damping, conditioned by liquid layer viscosity. As a result, during a short period of time transitional processes quickly go out and the stationary oscillations emerge [17]. Therefore, further we will focus on considering of stationary oscillations regime.

The dynamics of highly viscous incompressible liquid in a narrow channel (gap) is described by Navier-Stokes equations and continuity equation, in which inertia members are omitted [6, 18]

$$\frac{1}{\rho} \frac{\partial p}{\partial x} = \nu \left( \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_z}{\partial z^2} \right), \quad \frac{1}{\rho} \frac{\partial p}{\partial z} = \nu \left( \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_z}{\partial z^2} \right), \quad \frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z} = 0.$$  

where $p$ is the pressure, $\rho$, $\nu$ are the density and the kinematical coefficient of the liquid viscosity, $u_x$, $u_z$ are the liquid velocity vector projections on coordinates axis.
The equations (2) are expanded by the boundary conditions: liquid adhesion to narrow channel walls [7, 10]
\[
\begin{align*}
\frac{du}{dt} + u_x = 0 \quad \text{at} \quad z = 0, \quad u_x = 0, \quad u_z = 0 \quad \text{at} \quad z = \delta_0 + \theta x/\ell, \\
\end{align*}
\]
and the boundary conditions of its free edges leakage
\[
\begin{align*}
\rho = \rho^*(\theta x) \quad \text{at} \quad x = -\ell, \quad \rho = 0 \quad \text{at} \quad x = \ell.
\end{align*}
\]
Here \( x = x_m f(\theta x) \) is the law of the elastically fixed channel wall displacements, \( x_m \) is the oscillation amplitude of the elastically fixed channel wall.

The dynamic equation of elastically fixed wall is as follows
\[
md^2x/dt^2 + nx = N,
\]
where \( m \) is the wall mass, \( n \) is the elastic wall suspension rigidity, \( N \) is the force, acting from highly viscous liquid pulsating layer. The expression for force \( N \) takes the form
\[
N = \int_{-\ell}^{\ell} q_{xy} dx, \quad q_{xy} = \rho \nu (\partial u_z / \partial x + \partial u_x / \partial z) \quad \text{at} \quad z = 0,
\]
where \( q_{xy} \) is the shear stress, acting from liquid pulsating layer on elastically fixed channel wall.

**Solutions and discussions**

Let us introduce in our research the dimensionless variables
\[
\begin{align*}
\psi = \delta_0/\ell << 1, \quad \tau = \omega x, \quad \xi = x/\ell, \quad \zeta = z/\delta_0, \quad u_x = x_m \omega U_x, \quad u_z = x_m \omega U_z / \psi,
\end{align*}
\]
\[
\begin{align*}
p = x_m \rho \nu \omega (\delta_0 \psi)^2 P, \quad p^* = x_m \rho \nu \omega (\delta_0 \psi)^2 P^*, \quad \epsilon = \theta/\delta_0.
\end{align*}
\]
Here \( \psi \) is the small parameter, describing the problem.

Putting (7) in equations (2), boundary conditions (3), (4) and expression for force (6) we get
\[
\begin{align*}
\frac{\partial p}{\partial \xi} = \psi^2 \frac{\partial^2 U_x}{\partial \zeta^2} + \psi^2 \frac{\partial^2 U_z}{\partial \xi^2} \quad \frac{\partial p}{\partial \zeta} = \psi^2 \left[ \frac{\partial^2 U_x}{\partial \zeta^2} + \frac{\partial^2 U_z}{\partial \xi^2} \right], \quad \frac{\partial U_x}{\partial \xi} + \frac{\partial U_z}{\partial \zeta} = 0, \\
U_x = \psi \frac{df}{d\tau}, \quad U_z = 0 \quad \text{at} \quad \zeta = 0, \quad U_x = 0, \quad U_z = 0 \quad \text{at} \quad \zeta = 1 + \epsilon \xi, \\
P = P^* \quad \text{at} \quad \xi = -1, \quad P = 0 \quad \text{at} \quad \xi = 1, \\
N = \frac{b \ell \rho x_m \omega}{\delta_0 \psi} \int_{-\ell}^{\ell} \left( \psi^2 \frac{\partial U_x}{\partial \xi} + \frac{\partial U_z}{\partial \zeta} \right)_{\xi=0} d\xi.
\end{align*}
\]
Further, we note, that for a thin liquid layer members at $\psi^2$ in equations (8) and force (11) can be neglected in the setting under consideration. In results, we obtain the thin liquid layer hydrodynamic problem in the form of equations
\[
\frac{\partial P}{\partial \xi} = \partial^2 U_{\xi}/\partial \xi^2, \quad \partial P/\partial \zeta = 0, \quad \partial U_{\xi}/\partial \xi + \partial U_{\zeta}/\partial \zeta = 0,
\]
boundary conditions (9), (10) and expression for force
\[
N = \frac{h_0 \rho \nu \omega}{\delta \theta \nu} \int_{-1}^{1} \frac{\partial U_{\zeta}}{\partial \zeta} \bigg|_{\zeta=0} \, d\xi.
\]

The solution of the equations system (12) with boundary conditions (9), (10) are found in the form of
\[
U_{\zeta} = \varphi \frac{df}{d\tau} \left( 1 - \frac{\xi}{1 + \varepsilon} \right) - \frac{1}{2} \frac{\partial P}{\partial \zeta} \left( 1 + \varepsilon \xi - \zeta \right), \quad (14)
\]
\[
U_{\xi} = -\varphi \frac{df}{d\tau} \frac{1}{(1 + \varepsilon \xi)^2} \frac{\varepsilon \xi^2}{2} + \frac{1}{2} \frac{\partial^2 P}{\partial \xi^2} \left( \frac{\varepsilon^2}{2} (1 + \varepsilon \xi) - \frac{\varepsilon^3}{3} \right) + \frac{1}{2} \frac{\partial P \varepsilon \xi^2}{\partial \xi^2},
\]
\[
P = p^{*} \frac{(1 - \varepsilon)^2}{4} \frac{(1 - \xi)(2 + \varepsilon (\xi + 1))}{(1 + \varepsilon \xi)^2} + \frac{3 \varepsilon \varepsilon \xi (\varepsilon^2 - 1) df}{(1 + \varepsilon \xi)^2} \frac{d\tau}{d\tau}.
\]

It should be noted that
\[
\frac{\partial U_{\zeta}}{\partial \zeta} \bigg|_{\zeta=0} = -\varphi \frac{df}{d\tau} \left( \frac{1}{1 + \varepsilon \xi} + 3 \varepsilon \frac{\xi + \varepsilon}{(1 + \varepsilon \xi)^2} \right) + \frac{p^{*} (1 - \varepsilon)^2}{4 (1 + \varepsilon \xi)^2}.
\]

The equation (5) in consideration of (13), (15) and (7) take the form of
\[
m \frac{d^2 x}{dt^2} + 2K \frac{dx}{dt} + nx = p^{*} \omega^2 \frac{b \delta_0 (1 - \varepsilon^2)}{2},
\]
where
\[
2K = \frac{4 \ell \beta \rho \nu}{\delta_0} \left( \frac{1}{\varepsilon} \ln \left( \frac{1 + \varepsilon}{1 - \varepsilon} \right) - \frac{3}{2} \right).
\]

The solution of the equation (16) under the law of pressure change in time (1) takes the form of
\[
x = \frac{p_m b \delta_0}{2} \frac{1 - \varepsilon^2}{(n - m \omega^2)^2} \left[ (n - m \omega^2) \sin \omega t - 2K \omega \cos \omega t \right] = \frac{(1 - \varepsilon^2)b \delta_0/(2n) p_m A(\omega) \sin(\omega t + \varphi(\omega))}{(n - m \omega^2)^2 + 4K^2 \omega^2},
\]
\[
A(\omega) = 1/\sqrt{(1 - m \omega^2/n)^2 + (2K \omega/n)^2}, \quad \tan \varphi(\omega) = 2K \omega/(m \omega^2 - n),
\]
where $A(\omega)$ is the dimensionless amplitude frequency characteristic of elastically fixed tapered narrow channel wall, $\varphi(\omega)$ is the phase frequency characteristic of elastically fixed tapered narrow channel wall.

It should be noted that $A(\omega) \to 1$ at $\omega \to 0$ and $A(\omega) \to 0$ at $\omega \to \infty$.

Using the found expression for the wall displacements (17), we can finally define the law of pressure change dynamics along the channel in a dimensional form

$$p = p^* (\omega t) \frac{(1-\varepsilon^2) (1-(x/\ell))(2+\varepsilon((x/\ell)+1))}{4 (1+\varepsilon(x/\ell))^2} + p_m \Pi(\omega,x) \cos(\omega t + \varphi(\omega)),$$  

(19)

where $\Pi(\omega,x) = (3/2) \varepsilon(1-\varepsilon^2)((x/\ell)^2 - 1)(1+\varepsilon(x/\ell))^2 \rho v^2 b(\delta_p)^{-1} \omega A(\omega)$.

We make the following observations on this formula. The first component in the law of the pressure (19) reflects the edge pressure fall along the channel, and the second component represents the pressure in the liquid due to its shift by elastically fixed channel wall. It should be noted also that $\Pi(\omega,x) \to 0$ at $\omega \to 0$ and $\Pi(\omega,x) \to 0$ at $\omega \to \infty$.

Considering the special case of $\varepsilon \to 0$ (the channel is formed by two parallel walls) we obtain the following expressions for wall displacement and pressure

$$\tilde{x} = \frac{p_m b \delta_0}{2n} \frac{1}{\sqrt{1-m\omega^2/n}} \sin(\omega t + \arctg(2K\omega/(m\omega^2 - n))),$$  

(20)

$$\tilde{p} = p^* (\omega t) (1-(x/\ell))/2,$$  

(21)

where $2K = 2b \rho v / \delta_0$.

It is of interest to note that the pressure (21) have got only one component which presents the linear edge pressure fall along the channel.

The following dimensionless parameters and variable could be introduced $\Omega^2 = \eta/m$, $4D^2 = 4K^2/(mn)$ and $\eta = \omega/\Omega$, then expression for $A(\omega)$ (see (18)) take the form of

$$A(\eta) = 1/\sqrt{(1-\eta^2)^2 + 4D^2\eta^2}.$$  

(22)

Let us find maximum of $A(\eta)$, and differentiation of (22) gives at once

$$dA/d\eta = (2(1-\eta^2) - 4D^2/\eta) (1-\eta^2)^2 + 4D^2\eta^2)^{3/2}.$$  

(23)

After solving equation $(2-2\eta^2 - 4D^2)\eta = 0$ we obtain roots $\eta_1 = 0$, $\eta_2 = \sqrt{1-2D^2}$, $\eta_3 = -\sqrt{1-2D^2}$. The first root and the third one are physically impossible. The second root defines the maximum volume of $A(\eta)$ i.e. resonance oscillations [19]. Therefore the expression for resonance frequency takes the form of

$$\omega_{max} = \Omega \sqrt{1-2D^2} = \sqrt{n/m - 2K^2/m^2}.$$  

(24)
In order to illustrate the application of the proposed mathematical model, the channel with parameters: \( \ell = 0.1 \text{ m} \), \( \delta_0/\ell = 1/20 \), \( \ell/b = 1/20 \), \( \rho = 1.84 \times 10^3 \text{ kg/m}^3 \), \( \nu = 2.5 \times 10^{-4} \text{ m}^2/\text{s} \), \( m = 2.5 \text{ kg} \), \( n = 2 \times 10^3 \text{ kg/s}^2 \) are taken as example. The calculation results for \( A(\eta) \) are shown in fig. 2.

![Graph of dimensionless amplitude frequency characteristic \( A(\eta) \)](image)

**Conclusions**

The mathematical model of a tapered narrow channel with elastically fixed wall has been given for the vibration analysis of elastically fixed channel wall, interacting with pulsating highly viscous liquid layer. The numerical examples show that model can be of use for evaluation the possibilities of resonance oscillations formation taking into account the frequency range of possible pressure pulsating at the left channel edge. In frame of the mathematical model, we are found analytical expressions of channel wall displacements and pressure in highly viscous incompressible liquid layer. The amplitude frequency characteristic and phase frequency characteristic of elastically fixed tapered narrow channel wall are constructed and analytical expression for resonance frequency is obtained.

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**References**


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