Wavelet Based Numerical Solution of Second Kind Hypersingular Integral Equation

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Abstract

A Legendre multiwavelet based method is developed in this paper to solve second kind hypersingular integral equation by converting it into a Cauchy singular integro-differential equation. Multiscale representation of the singular and differential operators is obtained by employing Legendre multiwavelet basis. An estimate of the error of the approximate solution of the integral equation is obtained. A number of examples are given to illustrate the efficiency of the numerical method developed here.

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1. Introduction

Integral equations with singular kernels arise in various areas of mathematical physics. The singularity of the kernel may be weak or strong. When the singularity is weak (e.g. logarithmic type, Abel type), then the associated integral is defined in the sense of Riemann integral. If the singularity is strong, then the associated integral no longer exists in the sense of Riemann. However, the associated integral can be defined in some appropriate sense. For example, if the singularity is of Cauchy type, then it is defined in the sense of Cauchy principal value (CPV). If the singularity is stronger than Cauchy type, then also the associated integral can be defined in an appropriate manner. Singular integral equations, whose kernels have singularity stronger than Cauchy, are generally known as hypersingular integral equations, and the associated integral is defined in the sense of Hadamard finite part integral.


As application of fluid dynamics, Parsons and Martin [20, 21] formulated some water scattering problems involving thin straight vertical or inclined plates or thin curved plates completely submerged or partially immersed in infinitely deep water, in terms of hypersingular integral equations of first kind. The kernel of such a hypersingular integral equation has a hypersingular part and a non-singular smooth part. Parsons and Martin [20, 21] employed an expansion collocation method involving Chebyshev polynomials of the second kind to solve the hypersingular integral equations numerically and presented very accurate numerical estimates for quantities of physical interest for various geometrical configurations of the thin straight or curved plates. Later Midya et al.[19], Mandal and Gayen[15], Kanoria and Mandal[12] studied a number of water wave scattering problems using hypersingular integral equation formulations wherein the integral equations are of first kind with a kernel having a hypersingular part and a regular part. Chan et al.[8] demonstrated the applications of hypersingular integral equations of first kind to fracture mechanics in the theory of elasticity.

For solving hypersingular integral equations of second kind there exists several analytical as well as numerical methods in the literature. For example, the collocation and Galerkin methods were used by Iokimidis[11] to solve numerically a second kind hypersingular integral equation arising in elasticity for crack problems, the collocation method was used by Dragos[10] to solve Prandtl’s hypersingular integral
equation arising in aerodynamics, the complex variable method related to Riemann-Hilbert boundary value problem was used by Chakrabarti et al.\cite{5} to solve Prandtl’s equation in closed form. Mandal and Bera\cite{16} solved this Prandtl’s equation using a simple method based on polynomial approximation and obtained the exact solution. De Klerk\cite{9} earlier employed $L_p$-approximation method to solve hypersingular integral equation of second kind wherein the problem of solving the integral equation was formulated as solving a minimization problem.

In the collocation methods various types of orthogonal polynomials are used in the expansion of the unknown function of an integral equation. However, the approximate solution so obtained fails to provide local information such as smoothness or regularity of the solution. It may be possible that a function satisfying a singular integral equation may belong to the class $C^\alpha(0 < \alpha < 1)$ where $C^\alpha$ denotes the class of continuous function with Hölder exponent $\alpha(0 < \alpha < 1)$. It is thus desirable to search for an appropriate method which can provide the local information in the numerical solutions.

Multiresolution analysis (MRA) of function space in terms of refinable functions and wavelets with compact support has emerged as an efficient mathematical tool for analyzing phenomena arising in diverse fields of science and engineering. Numerical methods based on expansion in wavelet basis of some MRA of the space of square integrable functions defined over a finite domain have been employed successfully to obtain numerical solutions of weakly singular integral equations of second kind(cf. Paul et al.\cite{22}) which provide local informations about the function. This has also been the case for Cauchy singular integral equations of second kind as observed in the recent study by Paul et al.\cite{23}. The success of this study on weakly singular and Cauchy singular integral equations of second kind has motivated us to extend the method to hypersingular integral equations of second kind.

It may be noted that Alpert et al.\cite{1} and Alpert et al.\cite{2} are the pioneers in the development of generation of wavelets involving polynomials as refinable functions( called multiwavelets, also see Keinert\cite{13}), and used them to obtain approximate solution of second kind singular integral equations with logarithmic type kernel( weakly singular). Lakestani et al.\cite{14} employed Legendre multiwavelets to solve a weakly singular Fredholm integro-differential equation with Abel-type kernel while Paul et al.\cite{22, 23} employed Legendre multiwavelets to solve singular integral equations of second kind with Abel type kernel and Cauchy type kernel.

In this paper we consider Fredholm integral equations of second kind with hypersingular kernel, of the two forms as given by

$$a(x)u(x) + b(x) \int_0^1 \frac{u(t)}{(t - x)^2} dt = F(x)$$

(1.1a)
and
\[ a(x)u(x) + \int_0^1 \frac{b(t)u(t)}{(t-x)^2} dt = F(x) \quad (1.1b) \]

where the hypersingular integrals are defined in the sense of Hadamard finite part, \( a(x), b(x) \) and \( F(x) \) are known \( L^2([0,1]) \) functions and \( u(x) \) is an unknown whose solution is sought in the class of \( L^2([0,1]) \) functions. The paper is organised as follows. Reduction of the hypersingular integral equations (1.1a) and (1.1b) to equivalent Cauchy singular integro-differential equation is given in section 2. Properties of Legendre multiwavelets, their two scale relations are presented in section 3. The section 4 is concerned with multiscale approximation of a function and multiscale representation of the derivative operator in the Legendre multiwavelet basis. Representation of Cauchy singular integro-differential operator is given in section 5 while in section 6 representation of an operator \( A \) defined in section 2, is given. In section 7 the second kind Fredholm integral equation with hypersingular kernel is reduced to a system of linear equations and in section 8 error estimate for the approximate solution together with an estimate of local Hölder index is presented. A few illustrative numerical examples are given in section 9 while a short conclusion is given in section 10.

2. Reduction to Cauchy singular integro-differential equation

We consider the Fredholm integral equation of second kind with hypersingular kernel as given by equation (1.1a). Here the hypersingular integral is defined in the sense of Hadamard finite part of order 2 as given by
\[
\int_0^1 \frac{u(t)}{(t-x)^2} dt = \lim_{\epsilon \to 0} \left\{ \int_0^{x-\epsilon} \frac{u(t)}{(t-x)^2} dt + \int_{x+\epsilon}^1 \frac{u(t)}{(t-x)^2} dt - \frac{u(x-\epsilon) + u(x+\epsilon)}{\epsilon} \right\}, \quad 0 < x < 1. \quad (2.1)
\]

Following Boykov et al.[3], we can write
\[
\int_0^1 \frac{u(t)}{(t-x)^2} dt = -\frac{u(0)}{x} - \frac{u(1)}{1-x} + \lim_{\epsilon \to 0} \left\{ \int_0^{x-\epsilon} \frac{u'(t)}{(t-x)} dt + \int_{x+\epsilon}^1 \frac{u'(t)}{(t-x)} dt \right\}
\]
\[
= -\frac{u(0)}{x} - \frac{u(1)}{1-x} + \int_0^1 \frac{u'(t)}{t-x} dt, \quad 0 < x < 1
\]

where the integral in the right side of (2.2) is defined in the sense of CPV integral. Thus the hypersingular integral equation (1.1a) can be expressed as
\[(A_1u)(x) + (L_1Du)(x) = f(x), \quad 0 < x < 1. \quad (2.3)\]

where
\[(A_1u)(x) = x(1-x)a(x)u(x) - xb(x)u(1) - (1-x)b(x)u(0) \quad (2.4a)\]
Solution of hypersingular integral equation

\[(\mathcal{L}_1 u)(x) = \omega_1(x) \int_0^1 \frac{u(t)}{t-x} dt, \quad \omega_1(x) = x(1-x)b(x)\]  \hspace{1cm} (2.4b)

\[(\mathcal{D}u)(x) = u'(x)\] \hspace{1cm} (2.4c)

\[f(x) = x(1-x)F(x).\] \hspace{1cm} (2.4d)

It may be noted that (2.3) is a Cauchy singular integro-differential equation.

Similarly the integral equation (1.1b) can be reduced to another Cauchy singular integro-differential equation.

\[a(x)u(x) + \int_0^1 \frac{b(t)u(t)}{t-x} dt + \int_0^1 \frac{b'(t)u(t)}{t-x} dt - \frac{b(0)u(0)}{x} - \frac{b(1)u(1)}{1-x} = F(x), \] \hspace{1cm} (2.5)

if the function \(b(x)\) vanishes at the \(x = 0\) and \(x = 1\), then the Eq.(2.5) reduces to

\[a(x)u(x) + \int_0^1 \frac{b(t)u(t)}{t-x} dt + \int_0^1 \frac{b'(t)u(t)}{t-x} dt = F(x).\] \hspace{1cm} (2.6)

Thus the hypersingular integral equation (1.1b) involving \(b(x)\) vanishing at end points can be expressed as

\[(A_2u)(x) + (\mathcal{L}_2 \mathcal{D}u)(x) + (\mathcal{L}_3 u)(x) = F(x), \quad 0 < x < 1,\] \hspace{1cm} (2.7)

where

\[(A_2u)(x) = a(x)u(x)\] \hspace{1cm} (2.8a)

\[(\mathcal{L}_2 u)(x) = \int_0^1 \frac{\omega_2(t)u(t)}{t-x} dt, \quad \omega_2(x) = b(x)\] \hspace{1cm} (2.8b)

\[(\mathcal{D}u)(x) = u'(x)\] \hspace{1cm} (2.8c)

\[(\mathcal{L}_3 u)(x) = \int_0^1 \frac{\omega_3(t)u(t)}{t-x} dt, \quad \omega_3(x) = b'(x)\] \hspace{1cm} (2.8d)

\[f(x) = F(x).\] \hspace{1cm} (2.8e)

3. Legendre Multiwavelets

We use the Legendre polynomials \(P_0(x), P_1(x), ..., P_{K-1}(x)\) in the multiresolution analysis of \(L^2([0, 1])\) generated by the scale functions

\[\phi^i(x) := (2i + 1)^\frac{1}{2} P_i(2x - 1), \quad i = 0, 1, ..., K - 1; \quad 0 \leq x < 1.\] \hspace{1cm} (3.1)

and wavelets \(\psi^i(x)(i = 0, 1, ..., K - 1)\). At a particular resolution \(j\), these are given by

\[\phi^j_{i,k}(x) := 2^\frac{j}{2} \phi^i(2^j x - k), \quad j \in \mathbb{N} \cup \{0\}.\] \hspace{1cm} (3.1a)

\[\psi^j_{i,k}(x) := 2^\frac{j}{2} \psi^i(2^j x - k), \quad j \in \mathbb{N} \cup \{0\}.\] \hspace{1cm} (3.1b)

For a given \(j(> 0)\), translation is denoted by the symbol \(k\) \((k = 0, 1, ..., 2^j - 1)\). Clearly the support of \(\phi^j_{i,k}(x)\) and \(\psi^j_{i,k}(x)\) is \([\frac{k}{2^j}, \frac{k+1}{2^j}]\) for all \(i = 0, ..., K - 1\). The
functions $\phi_{j,k}^i(x)$ form an orthonormal basis for the approximation space $V_j^K$ in which the inner product is defined by

$$<f,g> = \int_0^1 f(x)g(x)dx.$$  

The two-scale relations among the scale functions between two consecutive scales $j$, $j + 1$ are given by

$$\phi_{j,k}^i(x) = \frac{1}{\sqrt{2}} \sum_{r=0}^{K-1} \left( h_{i,r}^0 \phi_{j+1,2k}^r(x) + h_{i,r}^{(1)} \phi_{j+1,2k+1}^r(x) \right)$$  \hspace{1cm} (3.2)

The wavelets $\psi_{j,k}^i(x)$, $(i = 0, 1, ..., K - 1; k = 0, 1, ..., 2^j - 1)$ for given $j$ can be found by using the relation

$$\psi_{j,k}^i(x) = \frac{1}{\sqrt{2}} \sum_{r=0}^{K-1} \left( g_{i,r}^0 \phi_{j+1,2k}^r(x) + g_{i,r}^{(1)} \phi_{j+1,2k+1}^r(x) \right)$$  \hspace{1cm} (3.3)

Here the symbols $H = \frac{1}{\sqrt{2}} \left( h^{(0)} : h^{(1)} \right)$ and $G = \frac{1}{\sqrt{2}} \left( g^{(0)} : g^{(1)} \right)$ with $h^{(s)} = [h_{i,r}^{(s)}], g^{(s)} = [g_{i,r}^{(s)}] (s = 0, 1)$ appearing in (3.2) and (3.3) are known as low- and high-pass filters for the MRA generated by Legendre multiwavelets involving $K$ vanishing moments of their wavelets. The low- and high-pass filters $H$ and $G$ and explicit forms of scale functions $\phi^l(x)$, wavelets $\psi^l(x)$ ($l = 0, 1, ..., K - 1$) for $K = 4$ and $K = 5$ can be found in Paul et al.[22].

4. Multiscale approximation  

4.1. Representation of a function. For a function $f \in L^2[0,1]$, the orthogonal projection of $f$ into the approximation space $V_j^K$ (the linear span of $\phi_{j,k}^i(x)$, $i = 0, 1, ..., K - 1; k = 0, 1, ..., 2^j - 1$) is

$$P_{V_j^K} : L^2[0,1] \rightarrow V_j^K$$

so that

$$f(x) \approx (P_{V_j^K} f)(x) \equiv \sum_{k=0}^{2^j-1} \sum_{i=0}^{K-1} c_{i,k}^j \phi_{j,k}^i(x).$$  \hspace{1cm} (4.1)

The decomposition

$$P_{V_0^K \oplus \bigoplus_{j=0}^{j=1} W_j^K} : L^2([0,1]) \rightarrow V_0^K \bigoplus \bigoplus_{j=0}^{j=1} W_j^K$$
of \( f(x) \) into the approximation space \( V^K_0 \) and detail spaces \( W^K_j (0 \leq j \leq J - 1) \) is given by

\[
\begin{align*}
f(x) & \approx f^{MS}_J(x) \\
& = P_{V^K_0} \left( \bigoplus_{j=0}^{J-1} W^K_j \right) f(x) \\
& = \sum_{i=0}^{K-1} \left\{ c^i_{j,0} \phi^i(x) + \sum_{j=0}^{J-1} \sum_{k=0}^{2^j-1} d^i_{j,k} \psi^i_{j,k}(x) \right\}
\end{align*}
\]

where \( c^i_{j,k} \) and \( d^i_{j,k} \) are the coefficients in the expansion of \( f(x) \) in the basis of the approximation space \( V^K_0 \) and detail space \( W^K_j \) respectively.

As in Paul et al.[22] the following notations are introduced for convenience.

\[
\Phi_{j,k}(x) = (\phi^0_{j,k}(x), \phi^1_{j,k}(x), \ldots \phi^{K-1}_{j,k}(x)),
\]

\[
\Psi_{j,k}(x) = (\psi^0_{j,k}(x), \psi^1_{j,k}(x), \ldots \psi^{K-1}_{j,k}(x)).
\]

The bases for approximation space \( V^K_j \), detail space \( W^K_j \) and \( \bigoplus_{j=0}^{J} W^K_j \) are then denoted by

\[
\Phi_J := (\Phi_{j,0}(x), \Phi_{j,1}(x), \ldots \Phi_{j,2^J-1}(x))
\]

which is a vector having \( 2^J K \) components,

\[
\Psi_J := (\Psi_{j,0}(x), \Psi_{j,1}(x), \ldots \Psi_{j,2^J-1}(x))
\]

which is a vector having \( 2^J K \) components, and

\[
j \Psi := (\Psi_0, \Psi_1, \ldots \Psi_J)
\]

which is a vector with \( (2^{J+1} - 1)K \) components. Also we introduce the symbols

\[
c_{j,k} := (c^0_{j,k}, c^1_{j,k}, \ldots c^{K-1}_{j,k}),
\]

\[
d_{j,k} := (d^0_{j,k}, d^1_{j,k}, \ldots d^{K-1}_{j,k}),
\]

\[
c_J := (c_{j,0}, c_{j,1}, \ldots c_{j,2^J-1})
\]

which is a vector with \( 2^J K \) components,

\[
d_J := (d_{j,0}, d_{j,1}, \ldots d_{j,2^J-1})
\]

which is vector with \( 2^J K \) components, and

\[
j d := (d_0, d_1, \ldots d_J)
\]

which is vector having \( (2^{J+1} - 1)K \) components. Then (4.1) and (4.2) can be expressed respectively as

\[
\left( P_{V^K_j} f \right)(x) = \Phi_J c_J^T
\]

(4.9)
and

\[ f(x) \approx f^{MS}(x) = \left( P^{\frac{J}{2}} \bigoplus_{j=0}^{J-1} W_{j}^{K} \right) f(x) = (\Phi_{0}, (J-1)\Psi) \left( \frac{c_{0}^{T}}{(J-1)d^{T}} \right) \]  (4.10)

4.2. Representation of the derivative of a function. Multiscale representation of the derivative of a function in Legendre multiwavelet basis is now given. Due to the finite discontinuity of the elements of Legendre multiwavelet basis in their domain, representation of the derivative is defined in the weak sense. Let \( D \left( \equiv \frac{d}{dx} \right) \) denote the derivative operator. In order to construct the representation of \( D \) in the Legendre multiwavelet basis, we consider the evaluation of integrals involving product of elements in the basis and their images under \( D \). Now we can write

\[ (D\phi^{l_{1}}) (x) = \sum_{l_{2}=0}^{K-1} \left( \rho_{D_{l_{2},l_{1}}} \phi^{l_{2}}(x) + \sum_{j_{2}=0}^{J-1} \sum_{k_{2}=0}^{2^{j_{2}}-1} \beta_{D_{l_{2},l_{1}}} (j_{2}, k_{2}) \psi_{j_{2},k_{2}}^{l_{2}}(x) \right), \]  (4.11)

\[ (D\psi^{l_{1}}_{j_{1},k_{1}}) (x) = \sum_{l_{2}=0}^{K-1} \left( \alpha_{D_{l_{2},l_{1}}} (j_{1}, k_{1}) \phi^{l_{2}}(x) + \sum_{j_{2}=0}^{J-1} \sum_{k_{2}=0}^{2^{j_{2}}-1} \gamma_{D_{l_{2},l_{1}}} (j_{2}, k_{2}; j_{1}, k_{1}) \psi_{j_{2},k_{2}}^{l_{2}}(x) \right), \]  (4.12)

where

\[ \rho_{D_{l_{1}l_{2}}} = \int_{0}^{1} \frac{d}{dx} \phi^{l_{1}}(x) \frac{d}{dx} \phi^{l_{2}}(x) \, dx, \]  (4.13)

\[ \alpha_{D_{l_{1}l_{2}}} (j_{2}, k_{2}) = \int_{0}^{1} \phi^{l_{1}}(x) \frac{d}{dx} \psi_{j_{2},k_{2}}^{l_{2}}(x) \, dx, \]

\[ \beta_{D_{l_{1}l_{2}}} (j_{1}, k_{1}) = \int_{0}^{1} \psi_{j_{1},k_{1}}^{l_{1}}(x) \frac{d}{dx} \phi^{l_{2}}(x) \, dx, \]

\[ \gamma_{D_{l_{1}l_{2}}} (j_{1}, k_{1}; j_{2}, k_{2}) = \int_{0}^{1} \psi_{j_{1},k_{1}}^{l_{1}}(x) \frac{d}{dx} \psi_{j_{2},k_{2}}^{l_{2}}(x) \, dx. \]

Since the integrals in (4.13) involve derivative of discontinuous functions, these are defined in the weak sense. Since the explicit forms of the Legendre multiwavelet basis functions are known, the integrals in (4.13) can be evaluated by splitting the integration domain at the points of discontinuity of the wavelets.
Thus the multiscale representation \((\Phi_0, (J-1)\Psi), D(\Phi_0, (J-1)\Psi)\) of \(D\) in the basis \((\Phi_0, (J-1)\Psi)\) can be written in the form

\[
D_{JM} = \begin{pmatrix}
\rho_D & \alpha_D(0) & \alpha_D(1) & \ldots & \alpha_D(J-1) \\
\beta_D(0) & \gamma_D(0,0) & \gamma_D(0,1) & \ldots & \gamma_D(0,J-1) \\
\beta_D(1) & \gamma_D(1,0) & \gamma_D(1,1) & \ldots & \gamma_D(1,J-1) \\
\vdots & \vdots & \vdots & \ldots & \vdots \\
\beta_D(J-1) & \gamma_D(J-1,0) & \gamma_D(J-1,1) & \ldots & \gamma_D(J-1,J-1)
\end{pmatrix}_{(2^J K) \times (2^J K)}
\]  

(4.14)

where

\[
\rho_D = \left[ \rho_{D_{1/2}} \right]_{K \times K},
\]

(4.15)

\[
\alpha_D(j) = \left[ \alpha_{D_{1/2}}(j,k) \right]_{K \times 2^j K},
\]

\[
\beta_D(j) = \left[ \beta_{D_{1/2}}(j,k) \right]_{2^j K \times K},
\]

\[
\gamma_D(j_1,j_2) = \left[ \gamma_{D_{1/2}}(j_1,k_1,j_2,k_2) \right]_{2^j K \times 2^j K}.
\]

Now

\[
D(\Phi_0, (J-1)\Psi) = (\Phi_0, (J-1)\Psi) \langle (\Phi_0, (J-1)\Psi), D(\Phi_0, (J-1)\Psi) \rangle = (\Phi_0, (J-1)\Psi) D_{JM}. 
\]

(4.16)

We now show that \(D_{JM}\) is a nilpotent matrix of order \(K\). From (4.16) we find

\[
D^K(\Phi_0, (J-1)\Psi) = (\Phi_0, (J-1)\Psi) \left( D_{JM} \right)^K. 
\]

(4.17)

Since the basis in the Legendre multiwavelet basis consists of piecewise continuous polynomials of degree at most \(K - 1\), all the elements of \(D^K(\Phi_0, (J-1)\Psi)\) are zero so that \(\left( D_{JM} \right)^K \equiv 0\).

5. Evaluation of integrals

Here the integrals in the multiscale representation of the operators \(L_1, L_2\) and \(L_3\) are evaluated.

5.1. Integrals involving scale functions. To evaluate the multiscale representation of \(L_1\) given in (2.4b) we use the notation

\[
\rho_{1;l_1l_2} = \int_0^1 \int_0^1 \frac{\omega_1(x) \phi_1^1(x) \phi_1^2(t)}{t - x} dt \, dx, \quad l_1, l_2 = 0, 1, \ldots, K - 1, 
\]

(5.2)

so that

\[
\rho_{1;l_1l_2} = \int_0^1 \omega_1(x) \phi_1^1(x) C_1^2(x) \, dx,
\]

where

\[
C_1^2(x) = \int_0^1 \frac{\phi_1^2(t)}{t - x} dt 
\]

(5.3)
The integrals in (5.4) can be evaluated explicitly if
\[
\omega = \begin{cases} 
\ln \frac{1}{x}, & \text{for } l_2 = 0, \\
\sqrt{3} (2 + (2x - 1)\ln \frac{1}{x}), & \text{for } l_2 = 1, \\
\sqrt{5} (6x - 3 + (6x^2 - 6x + 1)\ln \frac{1}{x}), & \text{for } l_2 = 2, \\
\frac{\sqrt{5}}{3} (60x^2 - 60x + 11 + (60x^3 - 90x^2 + 36x - 3)\ln \frac{1}{x}), & \text{for } l_2 = 3.
\end{cases}
\]
The values of \(\rho_{1,2}, l_1, l_2 = 0, 1, ..., K - 1(K = 4)\) can be obtained either by elementary integration or by using an appropriate quadrature rule depending on the form of \(b(x)\).

5.2. Integrals involving product of scale functions and wavelets. We denote
\[
\alpha_{1;2l_2}(j, k) = \int_0^1 \int_0^1 \frac{\omega_1(x) \phi^{l_1}(x) \psi^{l_2}_{j,k}(t)}{t - x} dt
\]
so that
\[
\alpha_{1;2l_2}(j, k) = 2^{\frac{l_1}{2}} \sum_{l_2=0}^{K-1} \left\{ g_{2l_2}^{(0)} \int_0^1 \omega_1(x) \phi^{l_1}(x) C^{l_2}(2^{j+l}x - 2k)dx + g_{2l_2}^{(1)} \int_0^1 \omega_1(x) \phi^{l_1}(x) C^{l_2}(2^{j+l}x - 2k - 1)dx \right\}
\] (5.4)

where \(C^{l_2}(x)\) is defined in (5.3), \(g_{2l_2}^{(0)}, g_{2l_2}^{(1)}\) are the components of the high-pass filter. The integrals in (5.4) can be evaluated explicitly if \(\omega(x)\) is a polynomial. If we denote
\[
\beta_{1;2l_2}(j, k) = \int_0^1 \int_0^1 \frac{\omega_1(x) \psi^{l_1}_{j,k}(x) \phi^{l_2}(t)}{t - x} dt \, dx,
\]
then
\[
\beta_{1;2l_2}(j, k) = \int_0^1 \omega_1(x) \psi^{l_1}_{j,k}(x) C^{l_2}(x) \, dx.
\] (5.5)

which can be evaluated explicitly if \(\omega_1(x)\) is a polynomial. It may be noted that if \(\omega(x)\) is not a polynomial, then the integrals in (5.4) and (5.5) can be evaluated numerically by using an appropriate quadrature rule.

5.3. Integrals involving product of wavelets. We denote
\[
\gamma_{1;2l_2}(j_2, k_1, j_2, k_2) = \int_0^1 \int_0^1 \frac{\omega_1(x) \psi^{l_1}_{j_1,k_1}(x) \psi^{l_2}_{j_2,k_2}(t)}{t - x} dt \, dx, \quad j_2 \geq j_1 \geq 0.
\]

Then using definition (3.1b) followed by transformation of variable one can obtain
\[
\gamma_{1;2l_2}(j_2, k_1, j_2, k_2) = \int_0^1 \int_0^1 \frac{\omega_1(x) \psi^{l_1}_{j_1,k_1}(x) \psi^{l_2}_{j_2,k_2}(t)}{r_1 + t - x} \, dt \, dx
\] (5.6)

where
\[
k_2 - 2^{j_2-j_1}k_1 = r_1 2^{j_2-j_1} + r_2, \quad r_1 \in \mathbb{Z}, \quad r_2 = 0, 1, ..., 2^{j_2-j_1} - 1.
\] (5.7)
For $r_1 \neq 0$, the integral in (5.6) can be evaluated by an appropriate quadrature rule, while for $r_1 = 0$, it can be written in the form

$$
\gamma_{i;1l_2}(j_1, k_1, j_2, k_2) = 2^{j_2-j_1} \sum_{l_3=0}^{K-1} \left\{ g_{l_2l_3}^{(0)} \int_0^1 \omega_1 \left( \frac{x+k_1}{2^{j_1}} \right) \psi_{j_1}(x) C^{l_3}(2^{j_2-j_1+1}x-2r_2) dx + g_{l_2l_3}^{(1)} \int_0^1 \omega_1 \left( \frac{x+k_1}{2^{j_1}} \right) \psi_{j_1}(x) C^{l_3}(2^{j_2-j_1+1}x-2r_2-1) dx \right\} (5.8)
$$

For $j_2 \leq j_1$, (5.6) is of the form

$$
\gamma_{i;1l_2}(j_1, k_1, j_2, k_2) = \int_0^1 \int_0^1 \frac{\omega_1(\frac{x+k_1}{2^{j_1}})\psi_{j_1}(x)\psi_{j_2}(t)}{r_1 + t - x} dt \ dx. \quad (5.9)
$$

where

$$
k_1 - 2^{j_1-j_2}k_2 = r_1 \cdot 2^{j_1-j_2} + r_2, \ r_1 \in \mathbb{Z}, \ r_2 = 0, 1, ..., 2^{j_1-j_2} - 1. \quad (5.10)
$$

Also, for $j_2 \leq j_1$, (5.8) is of the form

$$
\gamma_{i;1l_2}(j_1, k_1, j_2, k_2) = 2^{j_2-j_1} \sum_{l_3=0}^{K-1} \left\{ g_{l_2l_3}^{(0)} \int_0^1 \omega_1 \left( \frac{x+k_2}{2^{j_2}} \right) \psi_{j_1-j_2, r_2}(x) C^{l_3}(2x) dx + g_{l_2l_3}^{(1)} \int_0^1 \omega_1 \left( \frac{x+k_2}{2^{j_2}} \right) \psi_{j_1-j_2, r_2}(x) C^{l_3}(2x-1) dx \right\} (5.11)
$$

Thus all the integrals involved in the representation of $\mathcal{L}_1$ given in (2.4b) in Legendre multiwavelet basis are now evaluated.

In the representation of $\mathcal{L}_2$ and $\mathcal{L}_3$ given by (2.8b) and (2.8b), the quantities $\rho_{i;1l_2}$ $\alpha_{i;1l_2}(j, k)$, $\beta_{i;1l_2}(j, k)$, $\gamma_{i;1l_2}(j_1, k_1, j_2, k_2)$ for $i = 2, 3$ can be obtained by using the following formula.

$$
\rho_{i;1l_2} = - \int_0^1 \omega_i(x) \phi^{l_2}(x) C^{l_1}(x) \ dx, \quad (5.12a)
$$

$$
\alpha_{i;1l_2}(j, k) = - \int_0^1 \omega_i(x) \psi_{j,k}^{l_2}(x) C^{l_1}(x) \ dx. \quad (5.12b)
$$

$$
\beta_{i;1l_2}(j, k) = - 2^j \sum_{l_3=0}^{K-1} \left\{ g_{l_1l_3}^{(0)} \int_0^1 \omega_i(x) \phi^{l_2}(x) C^{l_3}(2^{j+1}x-2k) dx + g_{l_1l_3}^{(1)} \int_0^1 \omega_i(x) \phi^{l_2}(x) C^{l_3}(2^{j+1}x-2k-1) dx \right\}. \quad (5.12c)
$$

$$
(5.12d)$$
We now denote the matrices with their dimensions in all three cases involving integral operators $L_i (i = 1, 2, 3)$ as

$$
\begin{align*}
\rho_i & := [\rho_{i;12}]_{K \times K}, \\
\alpha_i (j) & := [\alpha_{i;12}(j, k)]_{K \times (2^2 K)}, \\
\beta_i (j) & := [\beta_{i;12}(j, k)]_{(2^2 K) \times K}, \\
\gamma_i (j_1, j_2) & := [\gamma_{i;12}(j_1, j_2, k_2)]_{(2^2 K \times 2^2 K)}.
\end{align*}
$$

(5.13)

5.4. Multiscale representation of $L_i$. To present the multiscale representation of integral operators $L_i$ given in (2.4b), (2.8b) and (2.8d), we write

$$
(L_i g^{i1})(x) = \sum_{l_2=0}^{K-1} \rho_{i;l_2} g^{i2}(x) + \sum_{j_2=0}^{J-1} \sum_{k_2=0}^{2^{j_2}-1} \beta_{i;l_21}(j_2, k_2) g^{i2}_{j_2, k_2}(x)
$$

(5.14)

and

$$
(L_i g^{i1}_{j_1, k_1})(x) = \sum_{l_2=0}^{K-1} \alpha_{i;l_21}(j_1, k_1) g^{i2}(x) + \sum_{j_2=0}^{J-1} \sum_{k_2=0}^{2^{j_2}-1} \gamma_{i;l_21}(j_2, k_2, j_1, k_1) g^{i2}_{j_2, k_2}(x)
$$

(5.15)

Using (5.14) and (5.15) the multiscale representation

$$
\langle (\Phi_0, (J-1) \Psi), L_i (\Phi_0, (J-1) \Psi) \rangle
$$
of $L_i$ in the basis $(\Phi_0, (j-1)\Psi)$ can be written in the form

$$L_i^{MS} = \begin{pmatrix} \rho_i & \alpha_i(0) & \alpha_i(1) & \ldots & \alpha_i(J-1) \\ \beta_i(0) & \gamma_i(0,0) & \gamma_i(0,1) & \ldots & \gamma_i(0,J-1) \\ \beta_i(1) & \gamma_i(1,0) & \gamma_i(1,1) & \ldots & \gamma_i(1,J-1) \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ \beta_i(J-1) & \gamma_i(J-1,0) & \gamma_i(J-1,1) & \ldots & \gamma_i(J-1,J-1) \end{pmatrix}_{(2^J K x 2^J K)}$$

(5.16)

where the sub-matrices $\rho_i, \alpha_i, \beta_i, \gamma_i$ are given in (5.13). Thus

$$L_i(\Phi_0, (j-1)\Psi) = (\Phi_0, (j-1)\Psi) L_i^{MS}.$$  

(5.17)

6. Evaluation of integrals in the representation of $A$

To obtain multiscale representation of the operator $A$ where $A = A_1$ is defined in (2.4a) and $A = A_2$ in (2.8a), we write

$$(A\phi^{j_1})(x) = \sum_{l_2=0}^{K-1} \sum_{j_2 = 0}^{J-1-2^{j_2-1}} \sum_{k_2 = 0}^{2^{j_2-1}} \beta_{A_{12l_1}}(j_2,k_2) \phi_{j_2,k_2}^{j_1}(x)$$

(6.1)

$$(A\psi^{j_1,k_1})(x) = \sum_{l_2=0}^{K-1} \sum_{j_2 = 0}^{J-1-2^{j_2-1}} \sum_{k_2 = 0}^{2^{j_2-1}} \gamma_{A_{12l_1}}(j_2,k_2,j_1,k_1) \psi_{j_2,k_2}^{j_1,k_1}(x)$$

(6.2)

where,

$$\rho_A = [\rho_{A_{11l_1}}, l_1, l_2 = 0, 1, \ldots, K-1]_{K x K}$$

with

$$\rho_{A_{11l_2}} = \int_0^1 \phi^{j_1}(x) (A\phi^{j_2}) (x) dx,$$

(6.3)

$$\alpha_A(j) = [\alpha_A(j,k), k = 0, 1, \ldots, 2^j-1]_{K x 2^j K}$$

with

$$\alpha_{A_{12l_1}}(j,k) = [\alpha_{A_{12l_1}}(j,k), l_1, l_2 = 0, 1, \ldots, K-1]_{K x K}$$

and

$$\alpha_{A_{12l_2}}(j,k) = \int_0^1 \phi^{j_1}(x) (A\psi^{j_2}_{j_2,k_2}) (x) dx,$$

(6.4)

$$\beta_A(j) = [\beta_A(j,k), k = 0, 1, \ldots, 2^j-1]_{2^j K x K}$$

with

$$\beta_{A_{12l_1}}(j,k) = [\beta_{A_{12l_1}}(j,k), l_1, l_2 = 0, 1, \ldots, K-1]_{K x K}$$

and

$$\beta_{A_{12l_2}}(j,k) = \int_0^1 \psi^{j_1}_{j_1,k_1}(x) (A\phi^{j_2}) (x) dx,$$

(6.5)

$$\gamma_A(j_1,j_2) = [\gamma_A(j_1,k_1;j_2,k_2), k_1 = 0, 1, \ldots, 2^{j_1}-1, k_2 = 0, 1, \ldots, 2^{j_2}-1]_{2^{j_1} K x 2^{j_2} K}.$$
with
\[ \gamma_A(j_1, k_1; j_2, k_2) = [\gamma_{A_{j_12}}(j_1, k_1; j_2, k_2), \ l_1, l_2 = 0, 1, \ldots, K - 1]_{K \times K} \]
and
\[ \gamma_{A_{j_12}}(j_1, k_1; j_2, k_2) = \int_0^1 \psi_{j_1, k_1}(x) (A\psi_{j_2, k_2})(x) \, dx. \]  \hspace{1cm} (6.6)

Then the multiscale representation \( \langle (\Phi_0, (J-1)\Psi) \rangle \) of \( A \) in the basis \( (\Phi_0, (J-1)\Psi) \) is
\[
A_{jMS} = \begin{pmatrix}
\rho_A & a_A(0) & a_A(1) & \cdots & a_A(J - 1) \\
\beta_A(0) & \gamma_A(0, 0) & \gamma_A(0, 1) & \cdots & \gamma_A(0, J - 1) \\
\beta_A(1) & \gamma_A(1, 0) & \gamma_A(1, 1) & \cdots & \gamma_A(1, J - 1) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\beta_A(J - 1) & \gamma_A(J - 1, 0) & \gamma_A(J - 1, 1) & \cdots & \gamma_A(J - 1, J - 1)
\end{pmatrix}_{(2^J \times 2^J K)}
\]  \hspace{1cm} (6.7)

Thus
\[ A(\Phi_0, (J-1)\Psi) = (\Phi_0, (J-1)\Psi) A_{jMS}. \]  \hspace{1cm} (6.8)

7. Solution of the Integral Equation

We seek solution in the class of \( L^2[0, 1] \) functions. Using the multiscale representation of \( u(x) \) (satisfying the hypersingular integral equation (1.1a) or (1.1b) or equivalently the Cauchy singular integro-differential equation (2.3) or (2.7)) as given by
\[ u(x) \approx u_{jMS}^T(x) = (\Phi_0, (J-1)\Psi) \begin{pmatrix} a_0^T \\ J^{-1}b^T \end{pmatrix}, \]  \hspace{1cm} (7.1)
and (4.10) in the equation (2.3), we obtain
\[ (A(\Phi_0, (J-1)\Psi) + L_1D(\Phi_0, (J-1)\Psi)) \begin{pmatrix} a_0^T \\ J^{-1}b^T \end{pmatrix} = (\Phi_0, (J-1)\Psi) \begin{pmatrix} c_0^T \\ J^{-1}d^T \end{pmatrix}. \]

Use (4.16), (5.17) and (6.8) followed by inner product with \( (\Phi_0, (J-1)\Psi)^T \) produces the linear system
\[ (A_{jMS}^T + L_{1MS}D_{jMS}) \begin{pmatrix} a_0^T \\ J^{-1}b^T \end{pmatrix} = \begin{pmatrix} c_0^T \\ J^{-1}d^T \end{pmatrix}. \]  \hspace{1cm} (7.2)

The components of \( a_0, (J-1)b \) are the coefficients of the multiscale expansion of the unknown solution at resolution \( J \) in the Legendre multiwavelet basis. The components of \( c_0, (J-1)d \) are the coefficients of the multiscale expansion of the known function \( f(x) \) at the same resolution \( J \) in (4.10) mentioned in section 4. Since the
Solution of hypersingular integral equation

matrix $A_j^{MS} + L_j^{MS}D_j^{MS}$ is well behaved, the unknown coefficients $a_0, (J-1)b$ can be obtained from

$$
\begin{pmatrix}
    a_0^T \\
    (J-1)b^T
\end{pmatrix} = (A_j^{MS} + L_j^{MS}D_j^{MS})^{-1} \begin{pmatrix}
    c_0^T \\
    (J-1)d^T
\end{pmatrix}.
$$

(7.3)

We can use the value of the components of $(a_0, (J-1)b)$ obtained in (7.3) to get the desired approximate solution.

8. ESTIMATION OF REGULARITY AND ERROR

An estimate for the local behaviour (regularity measured in terms of Hölder exponents) of the solution at any point in $[0, 1]$ can be obtained from a minor change in the formula given in Paul et al.[22]. We choose the dyadic interval $I_{j,k} = \left[\frac{k}{2^j}, \frac{k+1}{2^j}\right)$, and assume

$$u(x) \approx \text{constant} \left(x - \frac{k}{2^j}\right)^{\nu_{j,k}} \quad \text{for } x \in I_{j,k}.
$$

The proposed estimate of $\nu_{j,k}$ is given by

$$\nu_{j,k} \approx \begin{cases}
-\frac{1}{2} + \log_2 \left| \frac{d_{j+1,2k}}{d_{j+1,2k+1}} \right| & \text{for } \frac{k}{2^j} \leq x < \frac{k+1}{2^j}, \\
-\frac{1}{2} + \log_2 \left| \frac{d_{n,j}}{d_{n,j+2}} \right| & \text{for } \frac{k+1}{2^j} \leq x < \frac{k+1}{2^j}.
\end{cases}
$$

(8.1)

The $L^2$- error $\varepsilon_j^{L^2} = ||u - u_j^{MS}||_{L^2}$ in the approximate solution $u_j^{MS}$ given by (7.1) can be derived as in Paul et al.[22] and is given by

$$\varepsilon_j^{L^2} = \left[ \sum_{l=0}^{K-1} \sum_{j=J}^{\infty} \sum_{k=0}^{2^{j-1}-1} |b_{j,k}^l|^2 \right]^\frac{1}{2}.
$$

(8.2)

9. ILLUSTRATIVE EXAMPLES

In this section several numerical examples are given to illustrate the efficiency of this proposed method.

Example 1. Consider the integral equation (1.1a)

$$(2x - 1)^4u(x) + x(3 - 2x) \int_0^1 \frac{u(t)}{(t - x)^2} dt = F(x)$$

(9.1)

where

$$F(x) = (2x - 1)^4(1-\gamma+2\gamma x) - x(3 - 2x) \left\{ (1 - \gamma + 2\gamma x) \left( \frac{1}{1 - x} + \frac{1}{x} \right) - 2\gamma \ln \left| \frac{1 - x}{x} \right| \right\}$$

and $\gamma$ is a constant. The exact solution of (9.1) is given by[3]

$$u(x) = 1 - \gamma + 2\gamma x.$$
The Eq. (9.1) is of the form Eq.(1.1a) with
\[ a(x) = (2x - 1)^4, \ b(x) = x(3 - 2x) \]
so that it can be reduced to integro-differential equation (2.3) with appropriate forms of the coefficients involved in the operators \( D, R \) and \( A \) given by Eq. (2.4a)-(2.4d).

One can now use the representation of \( D, R \) and \( A \) given by (4.14),(5.16) and (6.7) to recast integro-differential equation for (9.1) to a system of linear algebraic equations for the unknown coefficients of \( a_0, \ b_{-1} \). Their values have been calculated for parameter \( \gamma = 0, 0.1, 0.5, 1, 5 \) at resolution \( J = 1 (K = 4) \) and presented in Table 1.

The multiscale approximation of the unknown solution \( u_{MS} \) obtained by using the coefficients in formula (4.2) have also been calculated and presented in the last column of Table 1 against the value of \( \gamma \) in each rows.

From the table it is found that all the wavelet coefficients of components of \( a_0 \) are zero and components of \( a_0 \) are found to coincide with the coefficients of the multiscale expansion of the exact solution \( u(x) \). Thus the present numerical method recovers the exact solution, as this method appears to be more efficient than the spline collocation method used by Boykov et al.[3] wherein errors exist even after choosing one thousand collocation points.

**Example 2.** We consider the integral equation
\[
    u(x) - \frac{1}{2\beta} \sqrt{x(1-x)} \int_0^1 \frac{u(t)}{(t-x)^2} dt = \frac{4\pi k}{\beta} \sqrt{x(1-x)}, \quad 0 < x < 1, \tag{9.2}
\]
with \( u(0) = 0 = u(1) \). This is known as the elliptic wing case of Prandtl’s equation([5,10,16]). Here \( \beta \) is a known constant and has the exact solution
\[
    u(x) = \frac{8k}{1 + \frac{2\beta^2}{\pi}} \sqrt{x(1-x)}.
\]
Solution of hypersingular integral equation

Comparing with (1.1a) we find

\[ a(x) = 1, \quad b(x) = -\frac{\sqrt{x(1-x)}}{2\beta}, \quad F(x) = \frac{4\pi k}{\beta} \sqrt{x(1-x)}. \]

Because of the end condition \( u(0) = 0, u(1) = 0 \), we can write

\[ u(x) = \sqrt{x(1-x)} v(x), \quad (9.3) \]

where \( v(x) \) is well-behaved unknown function of \( x \in (0, 1) \). Thus Eq.(9.2) reduces to

\[ v(x) - \frac{1}{2\beta} \int_0^1 \frac{\sqrt{t(1-t)} v(t)}{(t-x)^2} dt = \frac{4\pi k}{\beta}, \quad 0 < x < 1. \quad (9.4) \]

The above equation is of the form Eq.(1.1b) with

\[ a(x) = 1, \quad b(x) = -\frac{1}{2\beta} \sqrt{x(1-x)}, \quad F(x) = \frac{4\pi k}{\beta}. \]

Now applying the multiscale representation of (4.14),(5.16) and (6.7) the above integral equation is transformed into the system of linear algebraic equations. After solving the linear equations(in case of \( J = 1 \)) we find that all the components of \( b \) are zeros. The components of the coefficients of \( a \), \( b \) and the approximate solutions \( v_1(x)(J = 1) \) are presented in Table 2 for

\[ a) \quad k = 1 = \beta \quad b) \quad k = \frac{1}{2} = \beta \quad c) \quad k = 1 = 2\beta \quad a) \quad 2k = 1 = \beta \]

<table>
<thead>
<tr>
<th>( k )</th>
<th>( a^0_0 )</th>
<th>( a^1_0 )</th>
<th>( a^2_0 )</th>
<th>( b^0_0 )</th>
<th>( b^1_0 )</th>
<th>( b^2_0 )</th>
<th>( b^3_0 )</th>
<th>( \text{Approximate solution} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k = 1 = \beta )</td>
<td>( \frac{4\pi}{2\pi} )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( \frac{4\pi}{2\pi} )</td>
</tr>
<tr>
<td>( k = \frac{1}{2} = \beta )</td>
<td>( \frac{4\pi}{1 + \pi} )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( \frac{4\pi}{1 + \pi} )</td>
</tr>
<tr>
<td>( k = 1 = 2\beta )</td>
<td>( \frac{8\pi}{1 + \pi} )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( \frac{8\pi}{1 + \pi} )</td>
</tr>
<tr>
<td>( 2k = 1 = \beta )</td>
<td>( \frac{4\pi}{2 + \pi} )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( \frac{4\pi}{2 + \pi} )</td>
</tr>
</tbody>
</table>

Table 2. Components of \( a_0, b_0 \) and approximate solution for \( J = 1 \)

From this table it appears that the multiscale approximation(at resolution \( J = 1 \)) \( v_1(x) \) coincides with the exact solution \( v(x) \) irrespective of the choice of the parameter \( k \) and \( \beta \) involved in Eq.(9.3). After putting the value of \( v(x) \) in (9.3), we get the exact solution which is the same as given in Chakrabarti et al.[5] for the different values of \( k \) and \( \beta \).
Example 3. We have considered here the integral equation

\[ u(x) + \sqrt{x(1-x)} \int_0^1 \frac{u(t)}{(t-x)^2} dt = G(x), \quad 0 < x < 1, \tag{9.5} \]

where

\[ G(x) = \frac{\sqrt{x(1-x)}}{8} \left\{ 8x - 8x^2 + 3\pi(1 - 8x + 8x^2) \right\}. \]

The exact solution of Eq.(9.5) is given by

\[ u(x) = \left\{ x(1-x) \right\}^{\frac{3}{2}}. \tag{9.6} \]

The Eq.(9.5) is of the form (1.1a) with \( a(x) = 1, \quad b(x) = \sqrt{x(1-x)}. \) The integral equation is reduced to a system of linear equation by using the multiscale representation (4.14),(5.16) and (6.7) for \( K = 4, \quad J = 6. \) We evaluate the approximate solution \( u_4(x) \) and absolute error, which are presented in Fig 1. We have also estimated the Hölder exponent of the solution \( u(x) \) near the end points by using (8.1). The estimated values of \( \gamma_{j,0} \) (the exponent near \( x = 0 \)) and \( \gamma_{j,2J-1} \) (the exponent near \( x = 1 \)) are presented in Table 3 for \( j = 0, 1, 2, 3, 4. \) The sequences of \( \{\gamma_{j,0}, \quad j = 0, 1, 2, 3, 4\} \) and \( \{\gamma_{j,2J-1}, \quad j = 0, 1, 2, 3, 4\} \) in this table are found to be convergent and converges to 1.5. From the absolute error presented in Fig 1 we observe that the absolute error is comparatively high near the end points. This may be attributed due to the fractional form of the Hölder exponent near the end points.

\[ \text{Figure 1. The approximate solution } u_6^{MS}(x) \text{ and absolute error for } J = 6 \]

\[
\begin{array}{|c|c|c|}
\hline
j & \nu_{j,0} & \nu_{j,2J-1} \\
\hline
0 & 2.9 & 2.9 \\
1 & 1.94 & 1.94 \\
2 & 1.68 & 1.68 \\
3 & 1.58 & 1.58 \\
4 & 1.54 & 1.54 \\
\hline
\end{array}
\]

Table 3. Estimation of Hölder exponent of \( u(x) \) around the end points
To reduce the error near the end points in the numerical solution, \( u(x) \) can be expressed as
\[
u(x) = \sqrt{x(1-x)}v(x).
\] (9.7)

This representation will avoid the fractional part of the Hölder exponent near the end point for \( v(x) \). Now using (9.7) in (9.5), the integral equation is converted into
\[
v(x) + \int_0^1 \frac{\sqrt{t(1-t)}v(t)}{(t-x)^2} dt = \frac{1}{8} \left\{ 8x - 8x^2 + 3\pi(1 - 8x + 8x^2) \right\}.
\] (9.8)

The Eq.(9.8) is of the form (1.1b) with
\[
a(x) = 1, \quad b(x) = \sqrt{x(1-x)}.
\]

Now the integral equation reduces to a system of linear algebraic equation for \( K = 4, \ J = 1 \). By solving the linear equations we find that all the components of \( \alpha \) are zero. Therefore the approximate solution coincides with the exact solution \( v(x) = x(1-x) \). The components of \( a, \alpha \) and the approximate solution for \( v(x) \) are presented in Table 4.

<table>
<thead>
<tr>
<th>( a_0^0 )</th>
<th>( a_1^0 )</th>
<th>( a_2^0 )</th>
<th>( a_3^0 )</th>
<th>( b_0^0 )</th>
<th>( b_0^1 )</th>
<th>( b_0^2 )</th>
<th>( b_0^3 )</th>
<th>Approximate solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/π</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( x(1-x) )</td>
</tr>
</tbody>
</table>

**Table 4.** Components of \( a_0, \alpha \) and approximate solution of \( v_1(x)(J = 1) \) for \( J = 1 \) and Ex.3

From this table we find that the approximate solution of \( v(x) \) is
\[
v_1(x) = x(1 - x).
\]

Use of this result in (9.7) recovers the exact solution.

10. Conclusion

In this paper we have developed a numerical scheme based on multiresolution analysis of \( L^2([0,1]) \) functions generated by Legendre multiwavelets for solving hypersingular integral equation of second kind. In the process of development of the proposed scheme, the representation of derivative (\( \equiv \frac{d}{dx} \)) and the integro-differential operator (\( \equiv \int_0^1 \frac{d}{t-x} dt \)) with Cauchy singular kernel in the Legendre multiwavelet basis have been derived. These representations have been used to reduce the hypersingular integral equation to a system of algebraic equations which can be solved by any efficient numerical solver now available. An estimate of \( L^2 \)-error and local Hölder exponent of unknown solution in terms of wavelet coefficients have been presented. The efficiency of the proposed scheme has been tested by considering a few examples. This test shows that our scheme is very efficient.
References


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