Quasi-Optimal Reception of the Random Pulse with Arbitrary-Function Envelope and Unknown Time and Power Parameters

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Abstract
A new approach is introduced to the estimation of the fast fluctuating Gaussian pulses with arbitrary-function envelope. It helps us to obtain much simpler processing algorithms for random signals with unknown time and power parameters,
in comparison with available ones. The synthesis and analysis have been carried out of the quasi-optimal measurer of the high-frequency Gaussian random pulse signal with the time of arrival and the dispersion unknown. The asymptotically exact expressions are found for the estimation characteristics, including anomalous effects. By the methods of statistical computer modeling both the applicability and the efficiency of the considered technique for the measurement of the random pulsed signals in the conditions of the parametric prior uncertainty is corroborated, and we have the working capacity of the suggested measurer established and the applicability borders for the asymptotically exact formulas for its characteristics defined.

**Keywords:** fast fluctuating random signal, maximum likelihood method, decision statistics, unknown parameters, quasi-optimal estimation, local Markov approximation method, statistical modeling

1 Introduction

In a number of practical applications of a location and communication theory there is a need to measure the parameters of the random pulses received against hindrances [1, 2, etc.]. As an adequate mathematical model for such signals, the multiplicative combination

\[
x(t) = \xi(t) f[y(t - \lambda_0)] I\left(\frac{t - \lambda_0}{\tau}\right),
\]

is often used [3-5]. Here \(\lambda_0\) is the time of arrival, \(\tau\) is the duration of the signal, \(f(t)\) is the modulating function (envelope), which is describing the pulse form and is normalized so that \(\max f(t) = 1\), \(\gamma\) is a scale coefficient, and \(\xi(t)\) is the realization of the stationary centered Gaussian random process possessing the spectral density

\[
G(\omega) = (\pi D_0 / \Omega) \{ I[(\omega - \omega)/\Omega] + I[(\omega + \omega)/\Omega] \}.
\]

In Eq. (2) the designations are the following: \(\Omega\) is the band center, \(\Omega\) is the bandwidth, and \(D_0\) is the dispersion of the process \(\xi(t)\).

We presuppose that fluctuations \(\xi(t)\) are “fast”, that is the pulse duration \(\tau\) and the characteristic changing time \(\Delta t\) of the function \(f(t)\) essentially exceed the correlation time of the process \(\xi(t)\), so the following conditions are satisfied:

\[
\tau >> 2\pi / \Omega \quad (\mu = \tau \Omega / 2\pi >> 1), \quad \Delta t >> 2\pi / \Omega.
\]

In [4, 5] the problem is considered of the estimation of the time of arrival of the signal (1) observed against Gaussian white noise \(n(t)\) with one-sided spectral density \(N_0\), provided that all other pulse parameters are a priori known, or that
the pulse duration is inaccurately known. However, the dispersion $D_0$ of the process $\xi(t)$ can be unknown often enough and should be measured. So, it is of interest to find the structure and characteristics of the efficient measurer of the time of arrival of the signal (1) and its dispersion.

### 2 The Synthesis of the Estimation Algorithm

To obtain the estimate algorithm, we will use a maximum likelihood method [2, 6, 7], according to which it is necessary to form the decision statistics represented by the logarithm of the functional of the likelihood ratio (FLR) as the function $L(\lambda, D)$ of the current values $\lambda$ and $D$ of the unknown parameters $\lambda_0$ and $D_0$. If the conditions (3) are fulfilled, then, according to [4, 5], we have

$$ L(\lambda, D) = \frac{D}{N_0} M(\lambda, D) - \frac{\Omega}{2\pi} \int_{-\tau/2}^{\tau/2} \ln \left(1 + \frac{D}{E_N} f^2(\gamma t)\right) dt, $$

$$ M(\lambda, D) = \int_{-\tau/2}^{\tau/2} \frac{f^2(\gamma(t-\lambda))\gamma^2(t)}{E_N + Df^2(\gamma(t-\lambda))} dt, $$

where $E_N = N_0\Omega/2\pi$ is the average power of noise $n(t)$ within bandwidth of the process $\xi(t)$, and $\gamma(t) = \int_{-\infty}^{\infty} x(t') h(t-t')dt'$ is the output signal (response) of the filter with transfer function $H(\omega)$, which satisfies the condition $|H(\omega)|^2 = I[(\omega-\omega_0)/\Omega] + I[(\omega+\omega_0)/\Omega]$, on the observable data realization $x(t) = s(t) + n(t)$.

Then maximum likelihood estimates (MLEs) $\lambda_m$ and $D_m$ of the time of arrival $\lambda_0$ and dispersion $D_0$ of the random pulse (1) are determined as the position of the greatest maximum of decision statistics $L(\lambda, D)$:

$$ \lambda_m = \arg \sup_{\lambda \in [\Lambda_1, \Lambda_2]} L(\lambda, D_m), \quad D_m = \arg \sup_{D \geq 0} L(\lambda_m, D). $$

It is easy to see that the measurer (5) has multichannel structure and the infinitely many channels are required for its exact implementation that it is hardly probably in practice. In this connection, it might be useful to find single-channel quasi-optimal estimation algorithms of time and power parameters of the signal (1) close to the optimal algorithm (5) by their accuracy characteristics.

Similarly to [7, 8] it can be shown that MLE $\lambda_m \to \lambda_0$ is in mean square, if $\mu \to \infty$. Then, according to [8], the characteristics of MLE $D_m$ (5) coincide asymptotically (with increasing $\mu$) with the characteristics of the estimate
\[
D_{m0} = \arg \sup_{D \geq 0} L(\lambda_0, D).
\]

In general, the measurer (6) also has a multichannel organization, while the minimum variance \( V_{\min} \) of the estimate (6) determined by the Cramer-Rao formula [6] is equal to

\[
V_{\min} = \left( \frac{d^2}{dD^2} L(\lambda_0, D) \right)_{D=D_0}^{-1} = E_N^2 \left\{ \mu \int_{-1/2}^{1/2} \left[ f^2(\tilde{\gamma}t) d\tilde{t} \right]^2 \right\}^{-1}.
\]

Here \( q_0 = D_0/E_N \), \( \tilde{\gamma} = \gamma \tau \) and \( \tilde{t} = t/\tau \) is the normalized time. As it is noted in [6], the variance of MLE (6) coincides asymptotically (with output signal-to-noise ratio (SNR) increasing) with Eq. (7).

We introduce some simpler quasi-optimal estimate (QOE) of the dispersion \( D_{q0} \) instead of MLE \( D_{m0} \) (6). We carry out the synthesis of QOE \( D_{q0} \), on the assumption of the closeness of its variance to the minimum variance (7), provided that QOE can be technically implemented by single-channel units. Besides, in some limiting cases, QOE \( D_{q0} \) should converge to MLE \( D_{m0} \). As a result, we come to the estimate of the kind of

\[
D_{q0} = \max \left\{ 0, \left( M(\lambda_0) - \tau E_N \right)/F_2 \right\}.
\]

where \( y(t) \) is determined in the same way as in Eq. (4). This estimate possesses the following asymptotic characteristics

\[
b(D_{q0}|D_0) = 0, \quad v(D_{q0}|D_0) = E_N^2 \left\{ \int_{-1/2}^{1/2} \left[ 1 + q_0 f^2(\tilde{\gamma}t) \right] d\tilde{t} \right\}^{-1/2} \left( \mu \int_{-1/2}^{1/2} f^2(\tilde{\gamma}t) d\tilde{t} \right)^2.
\]

As it follows from Eqs. (7), (10), with the fulfillment of the conditions (3) the variance of the estimate \( D_{q0} \) (8) differs from the ultimate variance by no more than 5 % for a large class of modulating functions \( f(t) \). If \( f(t) \equiv 1 \), then variances (7) and (10) coincide, i.e. QOE (8) converges to MLE (6) as form of the modulating function \( f(t) \) approximates to the rectangular shape. It allows us to recommend the single-channel algorithm (8) instead of the more complex multichannel one (6) for the measurement of the dispersion of the pulse signal (1) in the practical applications, without significant loss in accuracy of the thus obtained estimate.
Quasi-optimal reception of random pulse ...

If the parameter $\lambda_0$ is unknown, then from Eq. (8) we obtain the estimate of the dispersion in the form of

$$D_q = \max \left[ 0, \left( M(\lambda_q) - \tau E_N / F_2 \right) \right],$$

(11)

where $\lambda_q = \arg \sup_{\lambda \in [\Lambda_1, \Lambda_2]} L(\lambda, D_q)$ is the estimate of the time of the pulse (1) arrival.

Substituting Eq. (11) in Eq. (4) and carrying out the optimization of the estimation algorithm in accordance with [9], we now get the estimate of the time of arrival

$$\lambda_q = \arg \sup_{\lambda \in [\Lambda_1, \Lambda_2]} M(\lambda)$$

(12)

instead of the estimate $\lambda_m$ (5). We also sign the estimates (11), (12) as QOEs. Indeed, if $f(t) \equiv 1$, then QOEs (11), (12) converge to the corresponding MLEs of the dispersion and the time of arrival of the high-frequency random pulse with rectangular envelope [10].

Fig. 1. The quasi-optimal measurer of the time of arrival and dispersion of the random pulse with arbitrary-function envelope

The measurer (11), (12) of the time and power parameters of the pulse signal (1) can be implemented as it is represented by the block diagram shown in Fig. 1. Here are the designations: 1 is the switch that is open for time $[\Lambda_1 - \tau / 2, \Lambda_2 + \tau / 2]$, 2 is a filter with transfer function $H(\omega) / \sqrt{F_2}$ (4), 3 is the squarer, 4 is an integrator, 5 is a delay line for the period $\tau$, 6 is the substracter, 7 is the retriever of the location of the input signal greatest maximum (extremator), 8 is the nonlinear element with the characteristic $x, 0_{\text{max}}$, 9 is the sampling device that at its output forms the input signal sampling from the instant $t = \lambda_q + \tau / 2$. 
3 The Characteristics of the Quasi-Optimal Estimation Algorithm

Let us find the characteristics of the estimates (11), (12). For this purpose we present the functional $M(\lambda)$ (9) as the sum of signal $S(l)$ and noise $N(l)$ functions [6, 7]:


Here $S(l) = \langle M(l) \rangle$, $N(l) = M(l) - \langle M(l) \rangle$, $l = \lambda / \tau$ is the normalized current value of the time of arrival, and the averaging $\langle \cdot \rangle$ is implemented through all the possible realizations $\hat{x}(t)$ with fixed values of the unknown parameters $\lambda_0$ and $D_0$. While executing the ratios (3), for signal function $S(l)$ we get

$$S(l) = A \int_{-1/2+\max(0,l-l_0)}^{1/2+\min(0,l-l_0)} f^2(\tilde{\tau}\tilde{t})d\tilde{t} + S_N, \quad S_N = \tau E_N,$$

where $A = \tau D_0$, $l_0 = \lambda_0 / \tau$, and $E_N$ is determined from Eq. (4).

In accordance with Eqs. (9), (13)

$$\langle N(l) \rangle = 0, \quad \sigma^2_{\text{max}} = \langle N^2(l) \rangle = \frac{\tau^2 E_N^2}{\mu} \int_{-1/2}^{1/2} \left[1 + q_0 f^2(\tilde{\tau}\tilde{t})\right]^2 d\tilde{t},$$

and

$$\langle N(l_1)N(l_2) \rangle = \frac{\tau^2 E_N^2}{\mu} \left\{ \max(0.1 - |l_1 - l_2|) + \int_{-1/2+\max(0,l_1-l_0,l_2-l_0)}^{1/2+\min(0,l_1-l_0,l_2-l_0)} \left[1 + q_0 f^2(\tilde{\tau}\tilde{t})\right]^2 d\tilde{t} \right\}.$$

During the analysis, we choose to divide all the estimates of the time of the pulse (1) arrival into the two classes: reliable and anomalous [4-7]. The estimate $l_q = \lambda_q / \tau$ is reliable, if it is within the interval limits $\Gamma_S = [l_0 - 1, l_0 + 1]$, where the signal function (14) is distinct from $S_N$. If QOE $l_q$ is out of the interval $\Gamma_S$, i.e. $l_q \in \Gamma_N = \Gamma \setminus \Gamma_S$, $\Gamma \equiv [\tilde{\Lambda}_1, \tilde{\Lambda}_2]$, $\tilde{\Lambda}_{1,2} = \Lambda_{1,2} / \tau$, then the estimate and the corresponding estimate error are designated as anomalous [6, 7].

It is necessary to consider the anomalous errors, in case where the length $m = \tilde{\Lambda}_2 - \tilde{\Lambda}_1$ of the prior interval $\Gamma$ of the possible values of the time of arrival $l_0$ is much greater than the length of the interval $\Gamma_S$ of the reliable estimate, i.e. the following condition holds

$$m >> 1.$$
According to [6, 7], while executing ratio (16), the conditional bias $b(l_q|l_0) = \langle l_q - l_0 \rangle$ and variance $V(l_q|l_0) = \langle (l_q - l_0)^2 \rangle$ of the estimate $l_q$, with the allowance for the anomalous errors, can be written down as follows:

$$b(l_q|l_0) = P_0 b_0(l_q|l_0) + (1 - P_0) \left[ (\bar{\lambda}_1 + \bar{\lambda}_2)/2 - l_0 \right].$$

$$V(l_q|l_0) = P_0 V_0(l_q|l_0) + (1 - P_0) \left[ (\bar{\lambda}_1^2 + \bar{\lambda}_2^2)/3 - l_0 (\bar{\lambda}_1 + \bar{\lambda}_2) + l_0^2 \right].$$

Here $b_0(l_q|l_0)$, $V_0(l_q|l_0)$, $P_0 = P[|l_q - l_0| \geq 1]$ are, correspondingly, the conditional bias, the conditional variance and the probability of the reliable estimate $l_q$ (12).

While determining $b_0(l_q|l_0)$, $V_0(l_q|l_0)$ and $P_0$, we are limited to the condition of a high posterior accuracy, when the output power SNR $\zeta^2$ of the algorithm (11), (12) is sufficiently great, i.e.

$$\zeta^2 = [S(l_0) - S_N]^2 / \langle N^2(l_0) \rangle = \mu q_0^2 \left[ \int_{-1/2}^{1/2} f^2(\tilde{\tau}) d\tilde{\tau} \right]^2 / \left[ \int_{-1/2}^{1/2} [1 + q_0 f^2(\tilde{\tau})] d\tilde{\tau} \right] \gg 1.$$  

(18)

The inequality (18) is valid with ratios (3) executed and when the value of $q_0$ is not too small. It is also presupposed that the function $f(\tilde{\tau})$ does not vanish in points $\tilde{\tau} = \pm 1/2$, i.e. the useful signal (1) is discontinuous [7].

Similarly to [7, 8], it can be shown that QLE $l_q$ converges in a mean square to a true value of the estimated parameter $l_0$ with the increase of $\zeta$. Hereupon, for the definition of the characteristics of the reliable estimate $l_q$ under $\zeta \gg 1$, it is sufficient to investigate the behavior of the functional $M(l)$ (9) in a close neighborhood of the point $l = l_0$. We designate

$$\Delta = \max \{|l_1 - l_0|, |l_2 - l_0|, |l_1 - l_2|\}.$$  

Then, taking into account the ratios (3) and with $\Delta \to 0$, the following asymptotic expansions appear to be true for Eqs. (14), (15):

$$S(l) = A \left[ \int_{-1/2}^{1/2} f^2(\tilde{\tau}) d\tilde{\tau} + \tilde{\tau}^2 \min(0, l - l_0) - \tilde{\tau}^2 (-\tilde{\tau}/2)\max(0, l - l_0) \right] + S_N + o(\Delta).$$

(19)
\[
\langle N(l_1)N(l_2) \rangle = \frac{\tau^2 E_N^2}{\mu} \left\{ \int_{-1/2}^{1/2} \left[ 1 + q_0 f^2(\tilde{\gamma} \tilde{\tau}) \right]^2 d\tilde{\tau} - |l_1 - l_2| + \left[ \left( 1 + q_0 f^2(\tilde{\gamma}/2) \right)^2 - 1 \right] \times \min(0, l_1 - l_0, l_2 - l_0) - \left[ 1 + q_0 f^2(-\tilde{\gamma}/2) \right]^2 - 1 \right] \max(0, l_1 - l_0, l_2 - l_0) \right\} + o(\Delta),
\]

(20)

where \( o(\Delta) \) denotes the higher-order infinitesimal terms compared with \( \Delta \).

Applying local Markov approximation method and its applications, as presented in [11, 12], and relying on the signal and noise functions properties (19), (20) of the decision statistics (9) for the conditional bias \( b_0(l_q | l_0) \), the conditional variance \( V_0(l_q | l_0) \) and the probability of the reliable estimate \( l_q \), we now get

\[
b_0(l_q | l_0) = 0, \quad V_0(l_q | l_0) = \frac{13 \left\{ 1 + [1 + q_0 f^2(\tilde{\gamma}/2)]^2 \right\} \left\{ 1 + [1 + q_0 f^2(-\tilde{\gamma}/2)]^2 \right\}}{8\mu^2 q_0^4 f^4(\tilde{\gamma}/2)f^4(-\tilde{\gamma}/2)}, \quad (21)
\]

\[
P_0 = \frac{z(\psi_+ + \psi_+)}{r} \int_1^\infty \exp \left\{ - \frac{m\kappa}{\sqrt{2\pi}} \exp \left( - \frac{\kappa^2}{2} \right) \right\} \times \left\{ \begin{array}{c}
\psi_- \exp \left[ \frac{\psi_-^2 z^2}{2} + \psi_- z \left( \frac{\kappa}{r} \right) \right] \Phi \left[ \frac{\kappa}{r} - z(\psi_- + 1) \right] + \\
+ \psi_+ \exp \left[ \frac{\psi_+^2 z^2}{2} + \psi_+ z \left( \frac{\kappa}{r} \right) \right] \Phi \left[ \frac{\kappa}{r} - z(\psi_+ + 1) \right] - \\
- \exp \left[ \frac{z^2 (\psi_- + \psi_+)^2}{2} + z(\psi_- + \psi_+) \left( \frac{\kappa}{r} \right) \right] \Phi \left[ \frac{\kappa}{r} - z(\psi_- + \psi_+ + 1) \right] \end{array} \right\} d\kappa,
\]

(22)

where

\[
r = \sqrt{\int_{-1/2}^{1/2} \left[ 1 + q_0 f^2(\tilde{\gamma} \tilde{\tau}) \right]^2 d\tilde{\tau}}, \quad \psi_{\pm} = \frac{2f^2(\pm\tilde{\gamma}/2)}{\int_{-1/2}^{1/2} \left[ 1 + q_0 f^2(\tilde{\gamma} \tilde{\tau}) \right]^2 d\tilde{\tau}}. \quad (23)
\]

The accuracy of the formulas (21) increases with \( \mu \) (3), \( z \) (18), as well as the accuracy of the formula (22) – with the increase of \( \mu \) (3), \( z \) (18), \( m \) (16).

If \( f(t) \equiv 1 \), then from Eqs. (21), (22) we obtain the known expressions for the characteristics of the maximum likelihood estimate \( \lambda_m \) (5) of the time of arrival of the random pulse (1) with rectangular envelope [11, 13].
Quasi-optimal reception of random pulse …

From [4] and Eq. (21) follows that the characteristics of the reliable QOE (12) coincide with the corresponding characteristics of MLE of the time of arrival of the signal (1) with a priori known other parameters. Besides, under \( z > 1.5 \ldots 2 \) the probability (22) practically coincides with the probability of the reliable estimate of the time of arrival of the signal (1) with a priori known other parameters found in [4]. Therefore, the measurer of the time of arrival (12) can be used, even when the form of modulating function is unknown, instead of the technically more complex measurers (5) and [4], requiring the greater prior information for their implementation. And losses in accuracy of the obtained estimate are insignificant when output SNR varies widely.

Then we consider the characteristics of QOE \( D_q \) (11). For that purpose, similarly to [10], we write down the distribution function of the random variable \( U = \left[ M(h_q) - S_N \right] / \sigma_{\text{max}} \), where \( \sigma_{\text{max}} \) is determined from Eq. (15), as

\[
F_U(x) = F_S(x) F_N(rx),
\]

if \( m \gg 1 \), and

\[
F_U(x) = F_S(x),
\]

if \( m \lessgtr 1 \) (order of unit or less). Here

\[
F_N(x) = \begin{cases} \exp \left[ -\frac{mx}{\sqrt{2\pi}} \exp \left( -\frac{x^2}{2} \right) \right], & x \geq 1, \\ 0, & x < 1, \end{cases}
\]

\[
F_S(x) = \Phi(x - z) - \exp \left[ \psi_+^2 z^2 / 2 + \psi_- z (z - x) \right] \Phi[ x - z (\psi_+ + 1) ] - \exp \left[ \psi_+^2 z^2 / 2 + \psi_- z (z - x) \right] \Phi[ x - z (\psi_- + 1) ] + \exp \left[ z^2 (\psi_- + \psi_+)^2 / 2 + (\psi_- + \psi_+)(z - x) \right] \Phi[ x - z (\psi_- + \psi_+ + 1) ],
\]

and \( r \) is defined the same as in Eq. (23). The distribution function \( F_q(x|D_0) = P[D_q < x] \) of the estimate \( D_q \) is linked to the function \( F_U(x) \) by the relation

\[
F_q(x|D_0) = \theta(x) F_U \left( \frac{zx}{D_0} \right), \quad \theta(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases}
\]

Taking into account Eq. (26), we get the expressions for conditional bias \( b(D_q|D_0) = \langle D_q - D_0 \rangle \) and variance \( V(D_q|D_0) = \langle (D_q - D_0)^2 \rangle \) of the estimate \( D_q \) (11):
We can analytically integrate Eqs. (27) in case when \( m \gtrsim 1 \) only. Using approximation (25) for the function \( F_U(x) \), for the characteristics of the estimate \( D_q \) (11) we obtain

\[
\begin{align*}
    b(D_q|D_0) &= D_0 \left[ \frac{1}{z^2 \psi_- (\psi_- + \psi_+)} \right] \Phi(z) - 1 + \frac{1}{z \sqrt{2\pi}} \exp \left( -\frac{z^2}{2} \right) + \\
    &\quad + \frac{1}{z^2 \psi_-} \exp \left[ z^2 \psi_- \left( \frac{\psi_-}{2} + 1 \right) \right] \left[ 1 - \Phi(z(\psi_- + 1)) \right] + \\
    &\quad + \frac{1}{z^2 \psi_+} \exp \left[ z^2 \psi_+ \left( \frac{\psi_+}{2} + 1 \right) \right] \left[ 1 - \Phi(z(\psi_+ + 1)) \right] - \frac{1}{z^2 (\psi_- + \psi_+)} \times \\
    &\quad \times \exp \left[ z^2 (\psi_- + \psi_+) \left( \frac{\psi_- + \psi_+}{2} + 1 \right) \right] \left[ 1 - \Phi(z(\psi_- + \psi_+)) \right],
\end{align*}
\]

\[ V(D_q|D_0) = D_0 \left[ 1 - \left[ 1 - \frac{1}{z^2} - 2 \left( \frac{2}{\psi_- (\psi_- + \psi_+)} \right) \right] \Phi(z) - \\
    - \frac{2}{z^2 \psi_-} \left( 1 - \frac{1}{z^2 \psi_-} \right) \exp \left[ z^2 \psi_- \left( \frac{\psi_-}{2} + 1 \right) \right] \left[ 1 - \Phi(z(\psi_- + 1)) \right] - \\
    - \frac{2}{z^2 \psi_+} \left( 1 - \frac{1}{z^2 \psi_+} \right) \exp \left[ z^2 \psi_+ \left( \frac{\psi_+}{2} + 1 \right) \right] \left[ 1 - \Phi(z(\psi_+ + 1)) \right] + \\
    + \frac{2}{z^2 (\psi_- + \psi_+)} \left[ 1 - \frac{1}{z^2 (\psi_- + \psi_+)} \right] \exp \left[ z^2 (\psi_- + \psi_+) \left( \frac{\psi_- + \psi_+}{2} + 1 \right) \right] \times \\
    \times \left[ 1 - \Phi(z(\psi_- + \psi_+ + 1)) \right] - \frac{1}{z \sqrt{2\pi}} \exp \left( -\frac{z^2}{2} \right). \tag{28}
\]

The accuracy of the formulas (24), (26), (27) increases with \( \mu, z, m \), as well as the accuracy of the formulas (28) increases with \( \mu \) and \( z \).

Formulas (24), (26), (27) and (28) become considerably simpler in case of large \( \mu \) (\( z \)) values when the probability of the anomalous error \( P_a = P[l_q \not\in \Gamma_S] = 1 - P_0 \) can be neglected while estimating the time of arrival \( l_0 \). Then we get
Thus, the estimate $D_q$ (11) is conditionally unbiased in asymptotics, and, according to Eqs. (10), (29), its variance coincides asymptotically (with increasing $\mu$ and $z$) with variance of the estimate $D_q^0$ (8). Assuming in Eqs. (24)-(28) that $f(t) \equiv 1$, we obtain the expressions for distribution function and characteristics of MLE of the dispersion of a rectangular random pulse (1) with the unknown time of arrival [10].

4 The Results of Statistical Modeling

In order to carry out the experimental functional test of the introduced quasi-likelihood measurer (11), (12) and to establish the applicability borders of the asymptotically exact formulas for its characteristics, we apply the statistical computer modeling of the estimation algorithm (11), (12). For the reduction of the computational burden, we presuppose that $\xi(t)$ is a narrow-band random process, i.e. the condition $\Theta >> \Omega$ is satisfied. It is allowed to present the response $y(t)$ of the narrow-band filter with pulse characteristic $h(t)$ (4) through its low-frequency quadratures and to form the normalized decision statistics $M(\lambda)$ (9) as the sum of the two independent random processes

$$M(\lambda) = \left[ M_1(\lambda) + M_2(\lambda) \right]/2, \quad M_i(\lambda) = \int_{\lambda - \tau/2}^{\lambda + \tau/2} y_i^2(t) dt, \quad y_i(t) = \int_{\lambda - \tau/2}^{\lambda + \tau/2} x_i(t') h_0(t - t') dt', \quad x_i(t) = \xi_i(t) f(t - \lambda_0) \left[ f(t - \lambda_0)/\tau \right] + n_i(t), \quad i = 1, 2.$$  

Here $\xi_i(t)$ and $n_i(t)$ are statistically independent centered Gaussian random processes with spectral densities $G_i(\omega) = (2\pi D_0/\Omega)I(\omega/\Omega)$ and $N_0$ accordingly, and the spectrum $H_0(\omega)$ of the function $h_0(t)$ satisfies the condition: $|H_0(\omega)|^2 = I(\omega/\Omega)$.

During modeling, we follow the technique described in [11] and form the samples of the random processes $y_i(t)$ with the step $\Delta t = 0.1\pi/\Omega$, and then – the samples of the random process $M(\lambda)$ with the step $\Delta \lambda = 0.01\pi$ for all $\lambda \in [\Lambda_1, \Lambda_2]$. Thus the relative mean square error of the step approximation of the continuous realizations of the process $M(\lambda)$ in terms of the generated discrete samples does not exceed 10 %. Further, according to Eqs. (11), (12) the normalized OQEs of the time of arrival and dispersion of the random pulse (1) are determined.

Some results of the statistical modeling for $l_0 = (\tilde{\Lambda}_1 + \tilde{\Lambda}_2)/2, \quad \tilde{\Lambda}_1 = 1/2$. 

$$h(D_q | D_0) \approx \frac{D_0 \left( \psi_-^2 + \psi_- \psi_+ + \psi_+^2 \right)}{z^2 \psi_- \psi_+ (\psi_- + \psi_+)} = 0, \quad V(D_q | D_0) \approx \frac{D_0^2}{z^2}. \quad (29)$$
\( \tilde{\chi}^2 = m + 1/2 \) and \( f(\tilde{r}) = \exp(-\tilde{r}^2) \), are shown in Figs. 2a-4a, while for \( f(\tilde{r}) = 1 - |\tilde{r}|/4 \) – in Figs. 2b-4b. Each experimental value in Figs. 2-4 is obtained as a result of the processing of no less than \( 10^4 \) realizations of the decision statistics \( M(\lambda) \). So, with probability of 0.9, the confidence intervals boundaries deviate from experimental values no more than for 10...15%.

In Figs. 2 the solid lines represent the dependences (17) of the normalized conditional variance \( \tilde{V}_l(q_0) = 12V_l(t_q t_0)/m^2 \) of QOE \( l_q \) (12) from the parameter \( q_0 \) (7), taking into account the anomalous errors, if \( m = 20 \). Here the analogous dependences (21) of the normalized variance \( \tilde{V}_{0l}(q_0) = 12V_0(t_q t_0)/m^2 \) of the reliable QOE \( l_q \) are also drawn by dashed lines. Curves 1 are calculated with \( \mu = 50, 2 \to 100, 3 \to 200 \). The experimental values for \( \mu = 50, 100 \) and 200 are denoted by rectangles, crosses, and rhombuses (for the variance \( \tilde{V}_l \) of the estimate \( l_q \) with the allowance for the anomalous errors), respectively, as well as by circles, triangles and asterisks (for the variance \( \tilde{V}_{0l} \) of the reliable estimate \( l_q \)).

In Fig. 3 the theoretical dependences (28) of the normalized conditional variance \( \tilde{V}_q = V(D_q |D_0)/E_N^2 \) of the estimated dispersion \( D_q \) (11) are traced by solid lines, when \( m = 1 \) and the estimate of the time of arrival \( l_q \) is reliable. In Fig. 4 the analogous dependences (24), (26), (27) are plotted, when \( m = 20 \) and the anomalous errors are possible while measuring the time of arrival. Curves 1 correspond to \( \mu = 50 \), curves 2 to \( \mu = 100 \), curves 3 to \( \mu = 200 \). Experimental values of the variances \( \tilde{V}_q \) in Figs. 3, 4 are shown by squares, crosses and rhombuses for \( \mu \) equal to 50, 100, 200, respectively.

![Fig. 2. The theoretical and experimental dependences of the normalized variance of the estimate of the time of arrival of the random pulse: a) bell-shaped, b) triangle](image-url)
Quasi-optimal reception of random pulse ...

Fig. 3. The theoretical and experimental dependences of the normalized variance of the estimate of the dispersion of the random pulse with the reliable estimate of the time of arrival: a) bell-shaped, b) triangle

Fig. 4. The theoretical and experimental dependences of the normalized variance of the dispersion estimate of a random pulse taking into account anomalous effects: a) bell-shaped, b) triangle

The obtained results lead us to the following conclusions. As is evident from Figs. 2, the theoretical dependences (17) for the characteristics of the estimate of the time of the pulse (1) arrival, with the allowance for the anomalous errors, approximate the experimental data in a satisfactory manner, at least, for \( m \geq 20 \), \( \mu \geq 50 \), \( z \geq 1.5 \ldots 2 \), \( (f_{\text{max}} - f_{\text{min}}) / \mu < 4 \cdot 10^{-3} \). Here \( f_{\text{min}} = \min f(t) \) and \( f_{\text{max}} = 1 \) are the minimum and the maximum values of the function \( f(t) \).

If the SNR is not too big (\( z \leq 4.5 \ldots 5.5 \)), then the threshold effects related to the anomalous errors occurrence should be taken into account while estimating the time of arrival. This leads to a jump-like (compared with the case of the relia-
ble estimate) increase in the variance of QOE $l_\theta$. As $q_0$ increases, when $z > 4.5...5.5$, the variance $\tilde{V}_l$ converges to the variance $\tilde{V}_{0_l}$, and the estimate $l_\theta$ becomes reliable with the probability close to 1. From Eqs. (17), (22) and Figs. 2 follows that the minimum (threshold) value of the parameter $q_0$, at which the influence of the anomalous errors on the accuracy of QLE of the time of arrival can be still neglected, decreases with $\mu$ and increases with $m$.

For $z < 2...3$ the theoretical dependences (21) of the variance $\tilde{V}_{0_l}$ of the reliable estimate (12) deviate from the experimental ones substantially, as they are found with the finite length of the interval $\Gamma_\Sigma$ neglected. Deviation of the theoretical dependences $V_0(l_\theta | l_0)$ (21), $V(l_\theta | l_0)$ (17) from the experimental data is observed under high SNR, when $q_0 > 2...3$ too. This is due to the fact that Eq. (21) for the variance $V_0(l_\theta | l_0)$ of the reliable estimate of the time of arrival is obtained with the sizes of the order of correlation time of the random process $\xi(t)$ neglected. Therefore, the calculating error in Eqs. (17), (21) becomes significant, as the normalized variance of QOE $l_\theta$ decreases to the size of order $\mu^{-2}$.

Formulas (24), (26), (27) and (28) for the characteristics of the dispersion estimate $D_\theta$ (11) with and without the influence of the anomalous errors on the accuracy of the estimation of the time of arrival $\lambda_\theta$ (12) well approximate the experimental data under $m \geq 20$ or $m \leq 1$ and $\mu \geq 50$, $z \geq 4...5$. If $z > 5$, so the probability of the anomalous error during estimating the time of arrival can be neglected, then the variances $V(D_\theta | D_0)$ of the estimate $D_\theta$ (11) calculated with the help of Eqs. (24), (26), (27) and Eq. (28) practically coincide. In case of $z > 6$, for the calculation of the characteristics of the dispersion estimate algorithm (11), it is possible to use the formulas (29), instead of Eqs. (28), without appreciable loss in accuracy.

5 Conclusion

The optimal measurers of the several unknown parameters of the random pulse signals, designed with the help of the traditional (maximum likelihood, Bayesian) approaches, are of a rather complex multichannel structure. In order to obtain the effective single-channel estimation algorithms, the technique based on the closeness of the accuracy of the formed estimates of the continuous signal parameters to the potential accuracy (Cramer-Rao bound) can be used. Application of the specified technique allows us to synthesize the quasi-optimal estimation algorithm of the time of arrival of the Gaussian pulse which is invariant to the dispersion of the useful signal and to the form of its modulating function. At the same time, the characteristics of quality rating of the introduced
Quasi-optimal reception of random pulse ...

algorithm are comparable with the corresponding characteristics of the technically more complex maximum likelihood estimation algorithm. To implement the quasi-optimal measurer of the dispersion of the random pulse with the unknown time of arrival, we need to know only the second moment (energy) of modulating function. The accuracy of the quasi-optimal dispersion estimate does not almost differ from the accuracy of the maximum likelihood dispersion estimate for a wide class of modulating functions. Conclusions and recommendations are valid, if the output signal-to-noise ratio is greater than 2...4.

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