Potential Energies and Potential-Energy Tensors
for Subsystems: General Properties

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Abstract

With regard to generic two-component systems, the theory of first variations of global quantities is reviewed and explicit expressions are inferred for subsystem potential energies and potential-energy tensors. Performing a conceptual experiment, a physical interpretation of subsystem potential energies and potential-energy tensors is discussed. Subsystem tidal radii are defined by requiring an unbound component in absence of the other one. To this respect, a few guidance examples are presented as: (i) an embedding and an embedded homogeneous sphere; (ii) an embedding and an embedded truncated, singular isothermal sphere where related centres are sufficiently distant; (iii) a homogeneous sphere and a Roche system i.e. a mass point surrounded by a vanishing atmosphere. The results are discussed and compared with the findings of earlier investigations.

Keywords: gravitation: Newtonian theory, virial theorem, cosmology: dark matter, galaxies: tidal radius, globular clusters: tidal radius

1 Introduction

Potential energies and potential-energy tensors are key ingredients for the application of the virial method [11] Chap. 2, that is essentially the method of


Large-scale celestial bodies e.g., galaxies and galaxy clusters, appear to be made of at least two subsystems which link only via gravitational interaction, where each component is distorted by the tidal force induced by the remaining one(s). On the other hand, large-scale celestial bodies can no longer be conceived as isolated and are often sufficiently close to exhibit tidal effects even in absence of accretion or merging.

The formulation of the virial theorem (implying the application of the virial method) to each subsystem separately yields a larger amount of information with respect to the system as a whole [1] [7] [20]. To this respect, different kinds of potential energies and potential-energy tensors can be defined e.g., [8], namely (i) self, related to the integration of the gravitational potential from the subsystem under consideration on the mass distribution of the subsystem under consideration; (ii) interaction, related to the integration of the gravitational potential from another subsystem on the mass distribution of the subsystem under consideration; (iii) tidal, related to the integration of the virial due to the gravitational force from another subsystem on the mass distribution of the subsystem under consideration; (iv) residual, which is merely the difference tidal minus interaction.

Potential energies and potential-energy tensors of subsystems can be used, among others, for a definition of tidal radius [10] [25], an interpretation of the fundamental plane of elliptical galaxies [26], a formulation of stellar system thermodynamics [27], and related first variations can be used for an application of d’Alembert’s principle involving the determination of virtual displacements [26] [27].

The current paper is restricted to two-component systems, without loss of generality in that multi-component systems can be conceived as the collection of all the pairs made of a selected subsystem and another one. With regard to a generic two-component system, investigation is devoted to the following points: explicit expression of first variations of potential energies and potential-energy tensors with the addition of physical interpretation, considered in Section 2 and 3, respectively; a global criterion for the definition of tidal radius, considered in Section 4, where a few guidance examples are presented. The discussion and the conclusion make the subject of Section 5 and 6, respectively. Further
details on a number of arguments mentioned in the text are shown in the Appendix.

2 First variations

2.1 General remarks

Let an unperturbed (collisional or collisionless) self-gravitating fluid be taken into consideration, filling the volume, $S_o$, at the time, $t$, and let $\Phi_o$ be a global parameter, depending on a local parameter, $Q_o$, as:

$$\Phi_o = \int_{S_o} Q_o(x_{o1}, x_{o2}, x_{o3}, t) \, d^3 S_o ; \quad (1)$$

where $d^3 S_o = dx_{o1} \, dx_{o2} \, dx_{o3}$ is the volume of an infinitesimal (unperturbed) fluid element.

If the fluid has occurred to be slightly perturbed at some initial time, $t_i < t$, a different evolution takes place from that time on, and the global parameter under consideration reads:

$$\Phi = \int_{S} Q(x_1, x_2, x_3, t) \, d^3 S ; \quad (2)$$

where $d^3 S = dx_1 \, dx_2 \, dx_3$ is the volume of an infinitesimal (perturbed) fluid element.

According to the parent paper [11] Chap. 2, §15, the first variation of the global parameter, $\Phi$, caused by the perturbation, is defined as:

$$\delta \Phi = \int_{S} Q(\vec{r}, t) \, d^3 S - \int_{S_o} Q_o(\vec{r}_o, t) \, d^3 S_o ; \quad (3)$$

where $\vec{r} \equiv (x_1, x_2, x_3)$, $\vec{r}_o \equiv (x_{o1}, x_{o2}, x_{o3})$, and the coordinates of perturbed fluid elements are related to their unperturbed counterparts by the transformation:

$$x_k = x_{ok} + \xi_k(\vec{r}_o, t) ; \quad (4a)$$

$$\left| \frac{x_k - x_{ok}}{x_{ok}} \right| \ll 1 ; \quad k = 1, 2, 3 ; \quad (4b)$$

or, in other words, the perturbed fluid is in linear regime.

A change of variables, defined by Eq. (4a), implies the Jacobian:

$$J(x_{o1}, x_{o2}, x_{o3}, t)$$

$$= \begin{vmatrix}
\frac{\partial x_1}{\partial x_{o1}} & \frac{\partial x_2}{\partial x_{o1}} & \frac{\partial x_3}{\partial x_{o1}} \\
\frac{\partial x_1}{\partial x_{o2}} & \frac{\partial x_2}{\partial x_{o2}} & \frac{\partial x_3}{\partial x_{o2}} \\
\frac{\partial x_1}{\partial x_{o3}} & \frac{\partial x_2}{\partial x_{o3}} & \frac{\partial x_3}{\partial x_{o3}}
\end{vmatrix}
= \begin{vmatrix}
1 + \frac{\partial \xi_1}{\partial x_{o1}} & \frac{\partial \xi_2}{\partial x_{o1}} & \frac{\partial \xi_3}{\partial x_{o1}} \\
\frac{\partial \xi_1}{\partial x_{o2}} & 1 + \frac{\partial \xi_2}{\partial x_{o2}} & \frac{\partial \xi_3}{\partial x_{o2}} \\
\frac{\partial \xi_1}{\partial x_{o3}} & \frac{\partial \xi_2}{\partial x_{o3}} & 1 + \frac{\partial \xi_3}{\partial x_{o3}}
\end{vmatrix} ; \quad (5)$$
which may safely be approximated as:

\[ J(x_{o1}, x_{o2}, x_{o3}, t) = 1 + \text{div} \vec{\xi} \; ; \]  

\[ (6) \]

where \( \vec{\xi} \equiv [\xi_1(\vec{r}_o, t), \xi_2(\vec{r}_o, t), \xi_3(\vec{r}_o, t)] \), to the first order in the displacement.

Accordingly, the first variation, \( \delta \Phi \), expressed by Eq. (3), reads:

\[ \delta \Phi = \int_{S_o} Q(\vec{r}_o + \vec{\xi}, t) (1 + \text{div} \vec{\xi}) \, d^3S_o - \int_{S_o} Q_o(\vec{r}_o, t) \, d^3S_o \]

\[ = \int_{S_o} Q(\vec{r}_o + \vec{\xi}, t) \, d^3S_o + \int_{S_o} Q(\vec{r}_o + \vec{\xi}, t) \, \text{div} \vec{\xi} \, d^3S_o - \int_{S_o} Q_o(\vec{r}_o, t) \, d^3S_o \; ; \]

which is equivalent to:

\[ \delta \Phi = \int_{S_o} (\Delta Q + Q \, \text{div} \vec{\xi}) \, d^3S_o \; ; \]

\[ (7) \]

where \( \Delta Q = Q(\vec{r}_o + \vec{\xi}, t) - Q_o(\vec{r}_o, t) \) is the Lagrangian change in the local parameter, \( Q \), consequent to the displacement, \( \vec{\xi} \) \[11\] Chap. 2, §13.

The particularization of Eq. (7) to the special case where the local parameter coincides with the density i.e. \( Q = \rho \), yields:

\[ \delta \Phi = \delta \int_{S_o} \rho \, d^3S_o = \int_{S_o} (\Delta \rho + \rho \, \text{div} \vec{\xi}) \, d^3S_o = 0 \; ; \]

\[ (8) \]

owing to mass conservation during the first variation \[11\] Chap. 2, §15.

The particularization of Eq. (7) to the special case where the local parameter is expressible as a product where a factor is the density and the other one is an additional local parameter i.e. \( Q' = \rho Q \), yields:

\[ \delta \Phi = \delta \int_{S_o} Q' \, d^3S_o = \delta \int_{S_o} \rho Q \, d^3S_o = \int_{S_o} [\Delta (\rho Q) + \rho Q \, \text{div} \vec{\xi}] \, d^3S_o \; ; \]

which reduces to:

\[ \delta \Phi = \delta \int_{S_o} \rho Q \, d^3S_o = \int_{S_o} \rho \Delta Q \, d^3S_o \; ; \]

\[ (9) \]

owing to Eq. (8).

The further restriction that the local parameter, \( Q \), is not intrinsic to a generic fluid element, such as pressure or density, but something which it assumes simply by virtue of its position, such as gravitational potential, allows the validity of the relation \[11\] Chap. 2, §13:

\[ \Delta Q = \sum_{k=1}^{3} \xi_k \frac{\partial Q}{\partial x_{ok}} \; ; \]

\[ (10) \]
and Eq. (9) takes the form:

$$\delta \Phi = \{ \delta \int_{S_o} \rho Q d^3 S_o = \int_{S_o} \rho \sum_{k=1}^{3} \xi_k \frac{\partial Q}{\partial x_{ok}} d^3 S_o \};$$

if the global parameter, \( \Phi \), is a vector, or a tensor, then a similar relation holds for the first variation of each component.

The generalization of Eq. (11) to the case where the local parameter is expressible as a product, two factors being densities calculated at different points, and a third factor being an additional local parameter (not intrinsic to a generic fluid element) which depends on both positions i.e. \( Q'(\vec{r}, \vec{r}', t) = \rho(\vec{r}, t) \rho(\vec{r}', t) Q(\vec{r}, \vec{r}', t) \), reads [11] Chap. 2, §15:

$$\delta \Phi = \{ \delta \int_{S_o} \int_{S_o} \rho(\vec{r}_o, t) \rho(\vec{r}_o', t) Q(\vec{r}_o, \vec{r}_o', t) d^3 S_o d^3 S_o' \}
\] = \int_{S_o} \int_{S_o} \rho(\vec{r}_o, t) \rho(\vec{r}_o', t) \sum_{k=1}^{3} \left[ \xi_k(\vec{r}_o, t) \frac{\partial Q}{\partial x_{ok}} + \xi_k(\vec{r}_o', t) \frac{\partial Q}{\partial x_{ok}'} \right] d^3 S_o d^3 S_o' \};$$

where \( d^3 S_o \), \( d^3 S_o' \), are infinitesimal volume elements on the top of the radius vector, \( \vec{r}_o \), \( \vec{r}_o' \), respectively. If the global parameter, \( \Phi \), is a vector, or a tensor, then a similar relation holds for the first variation of each component.

The generalization of Eq. (12) to the case where the local parameter is expressible as a product, two factors being densities calculated at different points of different subsystems, denoted as \( u, v \), respectively, and a third factor being an additional local parameter (not intrinsic to a generic fluid element) which depends on both positions i.e. \( Q'(\vec{r}_u, \vec{r}_v, t) = \rho_u(\vec{r}_u, t) \rho_v(\vec{r}_v, t) Q(\vec{r}_u, \vec{r}_v, t) \), reads:

$$\delta \Phi = \{ \delta \int_{S_{ou}} \int_{S_{ov}} \rho_u(\vec{r}_{ou}, t) \rho_v(\vec{r}_{ov}, t) Q(\vec{r}_{ou}, \vec{r}_{ov}, t) d^3 S_{ou} d^3 S_{ov} \}
\] = \int_{S_{ou}} \int_{S_{ov}} \rho_u(\vec{r}_{ou}, t) \rho_v(\vec{r}_{ov}, t) \sum_{k=1}^{3} \left[ \xi_k^{(u)}(\vec{r}_{ou}, t) \frac{\partial Q}{\partial x_{ok}^{(u)}} + \xi_k^{(v)}(\vec{r}_{ov}, t) \frac{\partial Q}{\partial x_{ok}^{(v)}} \right] d^3 S_{ou} d^3 S_{ov} \};$$

where \( d^3 S_{ou} \), \( d^3 S_{ov} \), are infinitesimal volume elements on the top of the radius vector, \( \vec{r}_{ou}, \vec{r}_{ov} \), respectively, and \( \vec{r}_{ov} \equiv (x_{o1}^{(w)}, x_{o2}^{(w)}, x_{o3}^{(w)}) \), \( w = u, v \). If the global parameter, \( \Phi \), is a vector, or a tensor, then a similar relation holds for the first variation of each component.

### 2.2 Potential-energy tensors for subsystems

Let an unperturbed (collisional or collisionless), two-component, self-gravitating fluid be taken into consideration, where the subsystems, denoted as \( i \) and \( j \), respectively, interact only gravitationally. In finding the first variations of global
parameters, let attention be restricted to the potential-energy tensors \[1\] \[7\] \[8\] \[11\] Chap. 2, § 10:

\[
(\Omega_u)_{pq} = -\frac{1}{2} \int_{S_u} \rho_u(\vec{r})(V_u)_{pq}(\vec{r}) \, d^3S_u \quad ;
\]

\[
(W_{uv})_{pq} = -\frac{1}{2} \int_{S_u} \rho_u(\vec{r})(V_u)_{pq}(\vec{r}) \, d^3S_u \quad ;
\]

\[
(V_{uv})_{pq} = \int_{S_u} \rho_u(\vec{r})x_p \frac{\partial V_v}{\partial x_q} \, d^3S_u \quad ;
\]

where \(u = i, j\); \(v = j, i\); \(d^3S_u = dx_1 \, dx_2 \, dx_3\); \((V_u)_{pq}\) and \(V_u\) are the tensor potential and the potential, respectively \[11\] Chap. 2, § 10:

\[
(\mathcal{V}_u)_{pq}(\vec{r}) = G \int_{S_u} \rho_u(\vec{r}) \left[ \frac{3}{2} \sum_{k=1}^{3} (x_k - x_k')^2 \right]^{-3/2} \, d^3S_u' \quad ;
\]

\[
V_u(\vec{r}) = G \int_{S_u} \rho_u(\vec{r}) \left[ \frac{3}{2} \sum_{k=1}^{3} (x_k - x_k')^2 \right]^{-1/2} \, d^3S_u' \quad ;
\]

\[
\frac{\partial V_u}{\partial x_p} = G \int_{S_u} \rho_u(\vec{r}) \frac{\partial}{\partial x_p} \left[ \frac{3}{2} \sum_{k=1}^{3} (x_k - x_k')^2 \right]^{-1/2} \, d^3S_u' \quad ;
\]

\[
= -G \int_{S_u} \rho_u(\vec{r}) \frac{x_p - x_p'}{3} \left[ \sum_{k=1}^{3} (x_k - x_k')^2 \right]^{3/2} \, d^3S_u' \quad ;
\]

where \(G\) is the constant of gravitation and \(d^3S_u' = dx_1' \, dx_2' \, dx_3'\).

The gravitational potential, \(V_u\), the potential self energy, \(\Omega_u\), the potential interaction energy, \(W_{uv}\), and the potential tidal energy, \(V_{uv}\), make the trace of their tensor counterparts \[1\] \[7\] \[8\] \[11\] Chap. 2, § 10:

\[
3 \sum_{r=1}^{3} \Psi_{rr} = \Psi \quad ; \quad \Psi = V_u, \Omega_u, W_{uv}, V_{uv} \quad ;
\]

for a formal demonstration, an interested reader is addressed to the above quoted parent papers.

Let the local parameter be taken equal to the integrand, without density factors, in the explicit expression of the potential self-energy tensor, defined by Eq. (14) via (17). Using Eq. (12), the first variation of the potential self-energy tensor after some algebra reads \[11\] Chap. 2, § 15:

\[
\delta(\Omega_u)_{pq} = -\int_{S_{ou}} \rho_u \sum_{k=1}^{3} \left[ \epsilon_{k}^{(u)} \frac{\partial(V_u)_{pq}}{\partial x_k^{(u)}} \right] \, d^3S_{ou} \quad ;
\]
and the trace of the above tensor, owing to Eq. (20), reads:

$$\delta \Omega_u = - \int_{S_{ou}} \rho_u \sum_{k=1}^{3} \left[ \xi_k^{(u)} \frac{\partial V_u}{\partial x_{k}^{(u)}} \right] d^3 S_{ou} \; ; \tag{22}$$

that is the first variation of the potential self energy.

Let the local parameter be taken equal to the integrand, without density factors, in the explicit expression of the potential interaction-energy tensor, defined by Eq. (15) via (17). Using Eq. (13), the first variation of the potential interaction-energy tensor after some algebra reads:

$$\delta(W_{uv})_{pq} = - \frac{1}{2} \int_{S_{ou}} \rho_u \sum_{k=1}^{3} \left[ \xi_k^{(u)} \frac{\partial (V_v)_{pq}}{\partial x_{k}^{(u)}} \right] d^3 S_{ou}$$

$$- \frac{1}{2} \int_{S_{ov}} \rho_v \sum_{k=1}^{3} \left[ \xi_k^{(v)} \frac{\partial (V_u)_{pq}}{\partial x_{k}^{(v)}} \right] d^3 S_{ov} \; ; \tag{23}$$

and the trace of the above tensor, owing to Eq. (20), reads:

$$\delta W_{uv} = - \frac{1}{2} \int_{S_{ou}} \rho_u \sum_{k=1}^{3} \left[ \xi_k^{(u)} \frac{\partial V_v}{\partial x_{k}^{(u)}} \right] d^3 S_{ou}$$

$$- \frac{1}{2} \int_{S_{ov}} \rho_v \sum_{k=1}^{3} \left[ \xi_k^{(v)} \frac{\partial V_u}{\partial x_{k}^{(v)}} \right] d^3 S_{ov} \; ; \tag{24}$$

that is the first variation of the potential interaction energy.

The sum of the first and the last term on the right-hand side of Eqs. (23) and (24) is symmetric with respect to the exchange of the indexes, $u$ and $v$, which makes the following relations hold:

$$\delta(W_{ij})_{pq} + \delta(W_{ji})_{pq} = - \int_{S_{oi}} \rho_i \sum_{k=1}^{3} \left[ \xi_k^{(i)} \frac{\partial (V_j)_{pq}}{\partial x_{k}^{(i)}} \right] d^3 S_{oi}$$

$$- \int_{S_{oj}} \rho_j \sum_{k=1}^{3} \left[ \xi_k^{(j)} \frac{\partial (V_i)_{pq}}{\partial x_{k}^{(j)}} \right] d^3 S_{oj} \; ; \tag{25}$$

$$\delta W_{ij} + \delta W_{ji} = - \int_{S_{oi}} \rho_i \sum_{k=1}^{3} \left[ \xi_k^{(i)} \frac{\partial V_j}{\partial x_{k}^{(i)}} \right] d^3 S_{oi}$$

$$- \int_{S_{oj}} \rho_j \sum_{k=1}^{3} \left[ \xi_k^{(j)} \frac{\partial V_i}{\partial x_{k}^{(j)}} \right] d^3 S_{oj} \; ; \tag{26}$$

and, in addition:

$$\delta(W_{ij})_{pq} = \delta(W_{ji})_{pq} \; ; \tag{27}$$

$$\delta W_{ij} = \delta W_{ji} \; ; \tag{28}$$
as expected from the symmetry of the potential interaction-energy tensors with respect to the exchange of the indexes, \(i\) and \(j\) e.g., [8].

Let the local parameter be taken equal to the integrand, without density factors, in the explicit expression of the potential tidal-energy tensor, defined by Eq. (16) via (19). Using Eq. (13), the first variation of the potential tidal-energy tensor after some algebra reads:

\[
\delta (V_{uv})_{pq} = + \int_{S_{ou}} \rho_u \sum_{k=1}^{3} \left\{ \xi_k^{(u)} \frac{\partial}{\partial x_k^{(u)}} \left[ x_p^{(u)} \frac{\partial V_v}{\partial x_q^{(u)}} \right] \right\} \, d^3S_{ou} \\
- \int_{S_{ov}} \rho_v \sum_{k=1}^{3} \left[ \xi_k^{(v)} \frac{\partial (V_u)_{pq}}{\partial x_k^{(v)}} \right] \, d^3S_{ov} \\
- \int_{S_{ov}} \rho_v \sum_{k=1}^{3} \left\{ \xi_k^{(v)} \frac{\partial}{\partial x_k^{(v)}} \left[ x_p^{(v)} \frac{\partial V_u}{\partial x_q^{(v)}} \right] \right\} \, d^3S_{ov} ; \tag{29}
\]

and the trace of the above tensor, owing to Eq. (20), reads:

\[
\delta V_{uv} = + \int_{S_{ou}} \rho_u \sum_{k=1}^{3} \sum_{r=1}^{3} \left\{ \xi_k^{(u)} \frac{\partial}{\partial x_k^{(u)}} \left[ x_r^{(u)} \frac{\partial V_v}{\partial x_r^{(u)}} \right] \right\} \, d^3S_{ou} \\
- \int_{S_{ov}} \rho_v \sum_{k=1}^{3} \left[ \xi_k^{(v)} \frac{\partial V_u}{\partial x_k^{(v)}} \right] \, d^3S_{ov} \\
- \int_{S_{ov}} \rho_v \sum_{k=1}^{3} \sum_{r=1}^{3} \left\{ \xi_k^{(v)} \frac{\partial}{\partial x_k^{(v)}} \left[ x_r^{(v)} \frac{\partial V_u}{\partial x_r^{(v)}} \right] \right\} \, d^3S_{ov} ; \tag{30}
\]

that is the first variation of the potential tidal energy.

The first term on the right-hand side of Eqs. (29) and (30) is related to the effect of the variation on \(u\) subsystem, while the other two terms are related to the effect of the variation on \(v\) subsystem. In addition, the sum of the first and the last term is antisymmetric with respect to the exchange of the indexes, \(u\) and \(v\), which makes the following relations hold:

\[
\delta (V_{ij})_{pq} + \delta (V_{ji})_{pq} = - \int_{S_{oi}} \rho_i \sum_{k=1}^{3} \left\{ \xi_k^{(i)} \frac{\partial (V_j)_{pq}}{\partial x_k^{(i)}} \right\} \, d^3S_{oi} \\
- \int_{S_{oi}} \rho_j \sum_{k=1}^{3} \left[ \xi_k^{(j)} \frac{\partial (V_i)_{pq}}{\partial x_k^{(j)}} \right] \, d^3S_{oi} ; \tag{31}
\]

\[
\delta V_{ij} + \delta V_{ji} = - \int_{S_{oi}} \rho_i \sum_{k=1}^{3} \left\{ \xi_k^{(i)} \frac{\partial V_j}{\partial x_k^{(i)}} \right\} \, d^3S_{oi} \\
- \int_{S_{oi}} \rho_j \sum_{k=1}^{3} \left[ \xi_k^{(j)} \frac{\partial V_i}{\partial x_k^{(j)}} \right] \, d^3S_{oi} ; \tag{32}
\]
and, in addition:

\[
\delta(V_{ij})_{pq} + \delta(V_{ji})_{pq} = \delta(W_{ij})_{pq} + \delta(W_{ji})_{pq} ; \quad (33)
\]

\[
\delta(Q_{ij})_{pq} + \delta(Q_{ji})_{pq} = 0 ; \quad (34)
\]

\[
\delta V_{ij} + \delta V_{ji} = \delta W_{ij} + \delta W_{ji} ; \quad (35)
\]

\[
\delta Q_{ij} + \delta Q_{ji} = 0 ; \quad (36)
\]

as expected from the symmetry of the potential interaction-energy tensors and the antisymmetry of the potential residual-energy tensors:

\[
(Q_{uv})_{pq} = (V_{uv})_{pq} - (W_{uv})_{pq} ; \quad (37)
\]

\[Q_{uv} = V_{uv} - W_{uv} ; \quad (38)\]

with respect to the exchange of the indexes, \(u\) and \(v\), which translates into the following relations:

\[
(W_{ij})_{pq} = (W_{ji})_{pq} ; \quad (39)
\]

\[
(Q_{ij})_{pq} = -(Q_{ji})_{pq} ; \quad (40)
\]

\[W_{ij} = W_{ji} ; \quad (41)\]

\[Q_{ij} = -Q_{ji} ; \quad (42)\]

for further details, an interested reader is addressed to the parent paper [8].

The combination of Eqs. (35) and (36) yields:

\[
\delta(V_{uv})_{pq} = \delta(W_{uv})_{pq} + \delta(Q_{uv})_{pq} ; \quad (43)
\]

\[
\delta V_{uv} = \delta W_{uv} + \delta Q_{uv} ; \quad (44)
\]

via Eqs. (37) and (38).

A similar result holds for the potential self-energy tensor of the whole system:

\[
\Omega_{pq} = (\Omega_i)_{pq} + (\Omega_j)_{pq} + (W_{ij})_{pq} + (W_{ji})_{pq} = (\Omega_i)_{pq} + (\Omega_j)_{pq} + (V_{ij})_{pq} + (V_{ji})_{pq} ; \quad (45)
\]

where the related first variation, according to Eqs. (12) and (21), after some algebra reads:

\[
\delta \Omega_{pq} = -\int_{S_o} \rho \sum_{k=1}^{3} \xi_k \frac{\partial V_{pq}}{\partial x_k} d^3 S_o \\
= -\int_{S_o} (\rho_i + \rho_j) \sum_{k=1}^{3} \left[ \xi_k^{(i)} \frac{\partial (V_{pq})^{(i)}}{\partial x_k} + \xi_k^{(j)} \frac{\partial (V_{pq})^{(j)}}{\partial x_k} \right] d^3 S_o ; \quad (46)
\]

\[
(47)\]
owing to the additivity of densities and tensor potentials. Splitting in four the last integral, and using Eqs. (21), (23), and (35), the final result is:

\[
\delta \Omega_{pq} = \delta (\Omega_i)_{pq} + \delta (\Omega_j)_{pq} + \delta (W_{ij})_{pq} + \delta (W_{ji})_{pq} = \delta (\Omega_i)_{pq} + \delta (\Omega_j)_{pq} + \delta (V_{ij})_{pq} + \delta (V_{ji})_{pq} ;
\]

(48)

and a summation over all the diagonal components yields:

\[
\delta \Omega = \delta \Omega_i + \delta \Omega_j + \delta W_{ij} + \delta W_{ji} = \delta \Omega_i + \delta \Omega_j + \delta V_{ij} + \delta V_{ji} ;
\]

(49)

which is the counterpart of Eq. (48), with respect to tensor traces.

3 Physical interpretation

In general, the virial theorem holds for potential and kinetic energies which are averaged over a sufficiently long time e.g., [5] [18] Chap. II, §10. Similarly, the tensor virial theorem holds for potential-energy and kinetic-energy tensor components which are averaged over a sufficiently long time. For sake of brevity, averaged values, \(< \Omega_u >, < W_{uw} >, < V_{uv} >, < T_u >\), shall be denoted as \(\Omega_u, W_{uw}, V_{uv}, T_u\), including related tensor components.

Aiming to a physical interpretation of potential energies and potential-energy tensors, let an isolated subsystem, \(u\), be first considered. Accordingly, the condition of virial equilibrium reads:

\[
\Omega_u + 2T_u = 0 ;
\]

(50)

and the total energy is:

\[
E_u = \Omega_u + T_u = -T_u = \frac{1}{2}\Omega_u ;
\]

(51)

in absence of tidal interaction.

If the subsystem is infinitely dispersed i.e. each particle is infinitely distant from each other, related energy changes are:

\[
\Delta \Omega_u = \Omega'_u - \Omega_u = -\Omega_u ;
\]

(52)

\[
\Delta E_u = E'_u - E_u = T_u - (\Omega_u + T_u) = -\Omega_u ;
\]

(53)

provided the kinetic energy is left unchanged, \(\Delta T_u = T'_u - T_u = 0\), where the prime denotes the final configuration.

Then the amount of work which must be done upon the subsystem in order to effect the above mentioned transition is:

\[
L_u = -\Delta E_u = \Omega_u ;
\]

(54)
where, in general, $L = -(E_F - E_i)$ is the work required for a transition from an initial state (energy, $E_i$) to a final state (energy, $E_F$), and $L < 0$ means work to be done, $L > 0$ work to be returned. According to Eq. (54), the potential self energy, $\Omega_u$, represents the amount of work which must be done upon the subsystem, $u$, in order to effect an infinite dispersion of the particles e.g., [22] Chap. III, §76.

As a second step, let two subsystems, $i$ and $j$ be considered. The condition of virial equilibrium for a generic subsystem, $u = i, j$, reads [1] [7] [20]:

$$\Omega_u + V_{uv} + 2T_u = 0 \ ; \quad (55)$$

and the total energy is:

$$E_u = \Omega_u + W_{uv} + T_u = -T_u - Q_{uv} \ ; \quad (56)$$

in presence of tidal interaction.

The kinetic energy, $T_u$, is in part macroscopic due to e.g., orbital motion of the centre of mass and systematic rotation, and in part microscopic due to random motions. Systematic translation of the centre of mass is ruled out by virial equilibrium, which implies motion of the subsystem within a limited region of space e.g., [5] [18] Chap. II, §10.

If the two subsystems are placed one infinitely distant from the other, leaving both the potential self energy, $\Omega_u$, and the kinetic energy, $T_u$, unaltered keeping the centre of mass at rest, related changes are:

$$\Delta''\Omega_u = \Omega''_u - \Omega_u = 0 \ ; \quad (57)$$
$$\Delta''W_{uv} = W''_{uv} - W_{uv} = -W_{uv} \ ; \quad (58)$$
$$\Delta''T_u = T''_u - T_u = 0 \ ; \quad (59)$$
$$\Delta''E_u = E''_u - E_u = -W_{uv} \ ; \quad (60)$$

where the subsystem is no longer in virial equilibrium and must necessarily readjust as:

$$\Omega'_u + 2T'_u = 0 \ ; \quad (61)$$
$$E'_u = \Omega'_u + T'_u = -T'_u = \frac{1}{2}\Omega'_u = E''_u \ ; \quad (62)$$

where no energy dissipation occurs. Then related changes occur. Then related changes are:

$$\Delta'\Omega_u = \Omega'_u - \Omega''_u = \Omega'_u - \Omega_u \ ; \quad (63)$$
$$\Delta'T_u = T'_u - T''_u = T'_u - T_u \ ; \quad (64)$$
$$\Delta'E_u = E'_u - E''_u = 0 \ ; \quad (65)$$
$$\Delta'\Omega_u + \Delta'T_u = 0 \ ; \quad (66)$$
where Eq. (66) holds via (62) and (65).

Finally, changes corresponding to the whole transition are:

\[ \Delta \Omega_u = \Delta'' \Omega_u + \Delta' \Omega_u = \Delta' \Omega_u ; \]  
\[ \Delta T_u = \Delta'' T_u + \Delta' T_u = \Delta' T_u = -\Delta' \Omega_u = -\Delta \Omega_u ; \]  
\[ \Delta E_u = \Delta'' E_u + \Delta' E_u = -W_{uv} ; \]  

where, on the other hand:

\[ \Delta E_u = E'_u - E_u = -T'_u + T_u + Q_{uv} = -\Delta T_u + Q_{uv} = \Delta \Omega_u + Q_{uv} ; \]  

and the combination of Eqs. (69) and (70) via (38) yields:

\[ \Delta \Omega_u = -W_{uv} - Q_{uv} = -V_{uv} ; \]  

in terms of the potential tidal energy.

According to Eq. (69), the potential interaction energy, \( W_{uv} \), represents the amount of work which must be done upon the subsystem, \( u \), as a whole, in order to recede up to an infinite distance from the subsystem, \( v \), keeping the centre of mass at rest and preserving virial equilibrium. In this view, the sentence [22] Chap. III, §76:

"Their sum \( W = W_{ij} + W_{ji} \) represents the exhaustion of potential energy, due to the fact that the two bodies are non infinitely far apart."

should be interpreted.

According to Eq. (71), the potential tidal energy, \( V_{uv} \), represents the change (regardless of the sign) in potential self energy that is necessary for \( u \) subsystem maintains virial equilibrium in absence of \( v \) subsystem, keeping the centre of mass at rest. For further details, an interested reader is addressed to Appendix A, where a conceptual experiment is performed.

The above considerations can be extended to tensor components, provided the work-tensor, \( L_{pq} = -(E_F)_{pq} - (E_I)_{pq} \), is defined, where \( (E_K)_{pq} \) is the total energy-tensor related to the initial (\( K = I \)) and final (\( K = F \)) state of an assigned transition, and the trace equals the related scalar work, \( L \).

In the special case of homeoidally striated ellipsoids e.g., [3] [9] let the subsystem, \( i \), be defined by an inner ellipsoid, \( 0 \leq r \leq R_i \), and let the subsystem, \( j \), be defined by an outer homeoid, \( R_i \leq r \leq R_j \), where \( r \) is the radial coordinate and \( R_i, R_j \), define the inner and the outer boundary, respectively, with regard to a selected direction. Owing to Newton’s theorem e.g., [3] the resulting gravitational force exerted on \( i \) from \( j \) is null i.e. the gravitational potential induced by \( j \) is constant for \( 0 \leq r \leq R_i \). Accordingly, \( V_{ij} = 0 \) via
Eq. (16) and, in addition, $W_{ij} = W_{ji}$ via Eq. (41), $Q_{ij} = -Q_{ji}$ via Eq. (42), which by use of Eq. (38) implies the following relations:

$$W_{ji} = W_{ij} = -Q_{ij} = Q_{ji};$$  \hspace{1cm} (72)

$$V_{ji} = W_{ji} + Q_{ji} = W_{ij} - Q_{ij} = 2W_{ij};$$  \hspace{1cm} (73)

where Eq. (73) discloses that the potential tidal energy, $V_{ji}$, is twice the work which must be done upon the inner ellipsoid in order to recede up to an infinite distance from the outer homeoid, according to an earlier investigation restricted to spherical symmetry [17].

The above considerations may be extended to potential-energy tensors, $(\Omega_{pq})_{ij}$, $(V_{uv})_{pq}$, $(W_{uv})_{pq}$, $(Q_{uv})_{pq}$, where Eqs. (50)-(73) can be translated to related tensor components.

4 Tidal radius

Tidal effects do not necessarily imply stripping, in that gravitational forces from different subsystems could exhibit a similar orientation. For instance, let the centre of mass of a spherical-symmetric galaxy lie within the nuclear star cluster, and let a test particle of unit mass be located on the cluster surface along the straight line joining the galaxy and cluster centre of mass. It is apparent the gravitational force from the galaxy and the cluster, acting on the above mentioned test particle, point along the same direction towards related centre of mass e.g., [6] which implies no tidal stripping from the cluster surface.

In presence of stripping, the tidal radius of a subsystem can be defined using either a local (i.e. involving force balance on a test particle e.g., [2] [6] [14] [29] [30]) or a global (i.e. involving energy balance on the whole subsystem e.g., [10] [24]) criterion. On the other hand, in absence of stripping, the tidal radius of a subsystem has necessarily to be defined via a global criterion [25] [26] [27].

In the special case of similar and similarly placed spheroids, the tidal radius for the inner component can be related to a special configuration where the kinetic energy, $2T_i = -\Omega_i - V_{ij}$, as a function of the major semiaxis, $a_i$, attains an extremum point (minimum) for fixed major semiaxis, $a_j$, and masses, $M_i$, $M_j$, provided the two subsystems interact only via gravitation and the virial theorem holds for each one [25] [26] [27]. For sufficiently steep density profiles, no extremum point occurs and no value can be assigned to the tidal radius. For further details, an interested reader is addressed to the above quoted parent papers.

Aiming to a general criterion which can be applied regardless of subsystem density profile and shape, a different attempt shall be exploited here. Let two subsystems interact only via gravitation and the virial theorem hold for each
one. In the general case where the subsystems are not concentric, a necessary condition for virial equilibrium is that related centres of mass move along orbits within a limited region of space, which implies kinetic energy is partly due to systematic (orbital at least) motions and partly to random motions. If orbits lie outside an equipotential surface, the virial theorem must be related to values averaged on a time, \( \tau \), largely exceeding the orbital period, \( \tau_{\text{orb}} \), and the notation has to be intended as \( \Phi_u = \langle \Phi_u >_\tau \), \( \Phi = \Omega, T \); \( \Psi_{uv} = \langle \Psi_{uv} >_\tau \), \( \Psi = W, V, Q \); \( \tau \gg \tau_{\text{orb}} \). For further details, an interested reader is addressed to specific textbooks e.g., [18] Chap. II, §10.

For sake of simplicity, it shall be intended in the following that subsystem centre of mass moves along a fictitious circular orbit where potential and kinetic energy equal related averaged values along the real orbit. With regard to the generic subsystem, \( u \), the condition of virial equilibrium and the total energy are expressed by Eqs. (55) and (56), respectively.

If the subsystem, \( v \), is instantaneously dispersed to infinite distance, the remaining one, keeping the centre of mass at rest, relaxes to a virialized configuration where the total energy, via Eqs. (55), (56), reads:

\[
E'_u = \Omega'_u + T'_u = \Omega_u + T_u = -V_{uv} - T_u;
\]

and the energy change amounts to \( \Delta E = E'_u - E_u = -W_{uv} \), conformly to Eq. (60). Keeping the centre of mass at rest implies conversion of translation kinetic energy into systematic either rotation or oscillation kinetic energy where the latter, in turn, implies conversion of systematic oscillation into random kinetic energy via violent relaxation [21].

The final state is bound or unbound according if \( E'_u < 0 \) or \( E'_u > 0 \), respectively. The limiting case, \( E'_u = 0 \), can be expressed as:

\[
\Omega_u = -T_u; \quad V_{uv} = -T_u; \quad \Omega_u = V_{uv};
\]

and the radius (intended as the distance from the centre of mass to the boundary along a selected direction), \( R^*_u \), for which Eq. (75) holds, is defined as tidal radius (along that direction) of \( u \) subsystem. While the extremum point of the kinetic energy, \( T_i \), as a function of the major semiaxis, \( a_i \), implies \( \Omega_i \approx V_{ij} \) in the special case of similar and similarly placed spheroids [25], \( \Omega_i = V_{ij} \) in general via Eq. (75).

In the special case of homogeneous spheres, one completely lying within the other, the potential self, interaction, tidal and residual energy are expressed as:

\[
\Omega_i = -\frac{3}{5} \frac{GM^2_i}{a_i}; \quad \Omega_j = -\frac{3}{5} \frac{GM^2_j}{a_j};
\]

\[
W_{ij} = -\frac{3}{5} \frac{GM^2_i m}{a_i y^2} \left( \frac{5}{4} y^2 - 1 - \frac{5}{12} y_0^2 \right); \quad W_{ji} = W_{ij};
\]
Potential energies and potential-energy tensors for subsystems

\[ V_{ij} = -\frac{3}{5} \frac{GM_i^2}{a_i} \frac{m}{y^3} \left( 1 + \frac{5}{3} y_0^2 \right) ; \quad V_{ji} = -\frac{3}{5} \frac{GM_i^2}{a_i} \frac{m}{y^3} \left( \frac{5}{2} y^2 - \frac{3}{2} - \frac{5}{2} y_0^2 \right) ; \quad (78) \]

\[ Q_{ij} = -\frac{3}{5} \frac{GM_i^2}{a_i} \frac{m}{y^3} \left[ \frac{5}{4} \left( 1 - y^2 \right) + \frac{25}{12} y_0^2 \right] ; \quad Q_{ji} = -Q_{ij} ; \quad (79) \]

\[ m = \frac{M_j}{M_i} ; \quad y = \frac{a_j}{a_i} ; \quad y \geq 1 ; \quad y_0 = \frac{R_0}{a_i} ; \quad 0 \leq y_0 \leq y - 1 ; \quad (80) \]

where the indexes, \( i, j \), label the embedded and the embedding sphere, respectively, \( M \) and \( a \) denote mass and radius, respectively, and \( R_0 \) is the distance between the centre of the embedding and the embedded sphere. For detailed calculations including potential-energy tensors, an interested reader is addressed to Appendix B.

Accordingly, Eq. (75) via (76) and (78) takes the form:

\[ \frac{m}{y^3} \left( 1 + \frac{5}{3} y_0^2 \right) = 1 ; \quad (81) \]
\[ \frac{m}{y^3} \left( \frac{5}{2} y^2 - \frac{3}{2} - \frac{5}{2} y_0^2 \right) = \frac{m^2}{y} ; \quad (82) \]

for \( i \) and \( j \) subsystem, respectively.

Related tidal radii are \( a_i^* = a_j/y_i^* \) and \( a_j^* = a_i y_j^* \), where \( y_i^*; y_j^* \), are positive real solutions of Eq. (81), (82), respectively, and \( y_0 \) can be expressed in terms of \( y \) as:

\[ y_0 = \zeta (y - 1) ; \quad 0 \leq \zeta \leq 1 ; \quad y \geq 1 ; \quad (83) \]

accordingly, Eqs. (81)-(82) translate into:

\[ 3y^3 - 5m\zeta^2 y^2 + 10m\zeta^2 y - (5\zeta^2 + 3)m = 0 ; \quad (84) \]
\[ [5(1 - \zeta^2) - 2m]y^2 + 10\zeta^2 y - (5\zeta^2 + 3) = 0 ; \quad (85) \]

in the special case of concentric spheres, \( \zeta = 0 \), the solutions of Eqs. (84) and (85) are:

\[ y_i^* = m^3 ; \quad m \geq 1 ; \quad (86) \]
\[ y_j^* = \left( \frac{3}{5 - 2m} \right)^{1/2} ; \quad 1 \leq m < \frac{5}{2} ; \quad (87) \]

owing to the condition, \( y \geq 1 \).

Turning to the general case, a third-degree equation, Eq. (84), and a second-degree equation, Eq. (85), have to be solved for assigned \( \zeta \). In the latter alternative, real solutions occur provided the discriminant is nonnegative, which is equivalent to:

\[ m \leq \frac{5}{2} \frac{3 + 2\zeta^2}{5 + 5\zeta^2} < \frac{5}{2} ; \quad 0 < \zeta \leq 1 ; \quad (88) \]
and the solution of Eq. (85) reads:

\[ y_j^* = \frac{-5\zeta^2 \pm \sqrt{[5(3 + 2\zeta^2) - 2m(3 + 5\zeta^2)]^{1/2}}}{5(1 - \zeta^2) - 2m} \]

(89)

where the condition, \( y \geq 1 \), implies the following inequality:

\[ 1 \leq m \leq \frac{5}{2} \left( \frac{3 + 2\zeta^2}{23 + 5\zeta^2} \right) \]

(90)

which defines the domain of the reduced mass, \( m \), in the case under discussion. For further details, an interested reader is addressed to Appendix C. In the special case of concentric spheres, \( \zeta = 0 \), Eq. (90) reads \( 1 \leq m < \frac{5}{2} \) according to Eq. (87). In the special case of tangent spheres, \( \zeta = 1 \), Eq. (90) reads

\[ 1 \leq m \leq \frac{25}{16}. \]

The reduced mass, \( 1/m = M_i/M_j \) and \( m = M_j/M_i \), as a function of the reduced tidal radius, \( 1/y_i^* = a_i^*/a_j \) and \( y_j^* = a_j^*/a_i \), can be inferred from Eq. (81) and (82), respectively, as:

\[ \frac{1}{m} = \frac{5\zeta^2}{3y_i^*} - \frac{10\zeta^2}{3(y_i^*)^2} + \left( \frac{5\zeta^2 + 1}{(y_i^*)^3} \right) \frac{1}{(y_i^*)^3} \]

(91)

\[ m = \frac{5(1 - \zeta^2)(y_i^*)^2 + 10\zeta^2y_j^* - (5\zeta^2 + 3)}{2(y_j^*)^2} \]

(92)

where, in particular, \( 1/m \to 0 \) as \( 1/y_i^* \to 0 \), \( 1/m = 1 \) as \( 1/y_i^* = 1 \), and \( m \to 5(1 - \zeta^2)/2 \) as \( y_j^* \to +\infty \), \( m = 1 \) as \( y_j^* = 1 \). The existence of an extremum point (maximum) at \( y_i^* = 1 + (3/5)(1/\zeta^2) \) can also be ascertained, where \( m = (5/2)(3 + 2\zeta^2)/(3 + 5\zeta^2) \). The special cases, \( \zeta = \ell/10 \), \( 0 \leq \ell \leq 10 \), \( \ell \) integer, are plotted in Fig. 1. An inspection of Fig. 1 shows the occurrence of oblique inflection points for values of \( \zeta \) sufficiently close to unity i.e. sufficiently large distance between the centre of the embedding and the embedded sphere. It is apparent the reduced tidal radius, \( 1/y_i^* \), can be defined for reduced masses within the range, \( 0 \leq 1/m \leq 1 \), with regard to the embedded sphere. Conversely, the reduced tidal radius, \( y_j^* \), can be defined for reduced masses within the range, \( 0 < m < 5/2 \), with regard to the embedding sphere.

5 Discussion

The above results, concerning explicit expression and physical interpretation of potential energies and potential-energy tensors, related variations, and a global criterion for the definition of the tidal radius, are restricted to two-component systems for simplicity. On the other hand, an extension can be done to multi-component systems by (a) dealing separately with all the pairs
Potential energies and potential-energy tensors for subsystems

Figure 1: The reduced mass, $1/m = M_i/M_j$, vs. the reduced tidal radius, $1/y_i^* = a_i^*/a_j$, $0 \leq 1/y_i^* \leq 1$, (bottom left box) and the reduced mass, $m = M_j/M_i$, vs. the reduced tidal radius, $y_j^* = a_j^*/a_i$, $y_j^* \geq 1$, for $\zeta = \ell/10$, $0 \leq \ell \leq 10$, $\ell$ integer. Curves from top to bottom relate to increasing $\zeta$ outside the box and to decreasing $\zeta$ inside the box. See text for further details.

made of a selected subsystem and one among the others, and (b) summing up the results due to the additivity of the gravitational potential and the tensor potential.

The basic idea is that considering each subsystem in virial equilibrium under the tidal action from the other(s) allows a larger amount of information with respect to the whole system [1] [7] [8] [20]. In this view, a physical interpretation of the potential interaction energy and potential tidal energy can shortly be stated as follows. Given two subsystems, $u$ and $v$, subjected to gravitation only, the potential interaction energy, $W_{uv} = W_{vu}$, represents the amount of work which must be done on $u$ as a whole, in order to recede up to an infinite distance from $v$ preserving virial equilibrium, and the potential tidal energy, $V_{uv} = W_{uv} + Q_{uv} = W_{vu} - Q_{vu}$, represents the change (regardless of the sign) in potential self energy, $\Delta \Omega_u$, that is necessary for $u$ maintains virial equilibrium in absence of $v$, in any case keeping the centre of mass at
It is widely accepted large-scale astrophysical objects are made of at least two components, such as visible baryonic (including leptons)-dark nonbaryonic matter, bulge-disk, bulge-halo, compact body-accretion disk, and so on. For concentric subsystems, a definition of tidal radius, necessarily in absence of stripping, could be highly rewarding e.g., [25] [26] [27]. To this respect, a guidance example is restricted to homogeneous spheres for simplicity but, on the other hand, allows a complete description of a subsystem completely lying within the other, where extreme situations are concentric spheres and tangent spheres, respectively. For instance, a description in terms of truncated, singular isothermal spheres can be expressed analytically only for sufficiently large distance between the centre of the embedding and the embedded sphere [4] [10].

The presence of a nuclear star cluster in the Galaxy e.g., [13] [17] and similar or less massive galaxies e.g., [15] invokes a natural application of the criterion exploited in the current paper for the definition of tidal radius, extended to globular clusters. To this aim, three models shall be discussed, namely (i) homogeneous spheres; (ii) truncated, singular isothermal spheres; in both cases, one completely lying within the other, and (iii) a heterogeneous sphere completely lying within a Roche system i.e. a mass point surrounded by a vanishing atmosphere. The subsystems, representative of a globular cluster and the Galaxy, shall be denoted as \( i = C \) and \( j = G \), respectively.

Concerning homogeneous spheres, the plot of the reduced mass, \( \frac{1}{m} = \frac{M_C}{M_G} \), vs. the reduced tidal radius, \( \frac{1}{y_C} = \frac{a_C}{a_G} \), is shown in the bottom left box of Fig. 1 and zoomed in Fig. 2 (\( 0 \leq \frac{1}{m} \leq 1 \)), where the lower curve (\( \zeta = 0 \)) represents concentric spheres i.e. the nuclear star cluster, while higher curves (\( \zeta > 0 \)) represent increasingly distant globular clusters up to a tangential configuration (\( \zeta = 1 \)) with respect to the Galaxy.

Concerning truncated, singular isothermal spheres, the reduced mass as a function of the reduced tidal radius is expressed as:

\[
\frac{1}{m} = \frac{1}{y_C^*} ;
\]

which is acceptable to a good extent for \( y > y_0 \gg 1 \), or \( 0 \ll \zeta \leq 1 \). For further details, an interested reader is addredded to Appendix D. The dependence of the reduced mass, \( 1/m \), on the reduced tidal radius, \( 1/y_C^* \), is linear and independent on \( \zeta \), as shown in Fig. 2 (dashed line).

Concerning a heterogeneous sphere completely lying within a Roche system (mass point surrounded by a vanishing atmosphere), the reduced mass as a function of the reduced tidal radius is expressed as:

\[
\frac{1}{m} = \frac{1}{\nu_1} \frac{1}{y_C} \left( 1 - \frac{1}{y_C^*} \right)^{-1} ; \quad 0 \leq \frac{1}{y_C^*} \leq \frac{\zeta}{1+\zeta} ; \quad 0 \leq \zeta \leq 1 ; \quad (94)
\]
where $\nu_\Omega$ is a factor which depends on the density profile within the sphere, with regard to an external mass point, and:

$$\frac{1}{m} = \frac{5}{3} \left[ \frac{3}{2} - \frac{1}{2} \zeta^2 \left( \frac{1}{y_C^*} \right)^{-2} \left( 1 - \frac{1}{y_C^*} \right)^2 \right]; \quad 1 \geq \frac{1}{y_C^*} > \frac{\zeta}{1 + \zeta}; \quad 0 \leq \zeta \leq 1; \quad (95)$$

in the special case of a homogeneous sphere ($\nu_\Omega = 3/5$), with regard to an internal mass point. The functions, expressed by Eqs. (94) and (95), join at $1/y_C^* = \zeta/(1 + \zeta)$ as $1/m = 5/3$. For further details, an interested reader is addressed to Appendix E.

The dependence of the reduced mass, $1/m$, on the reduced tidal radius, $1/y_C^*$, is shown in Fig. 2 ($0 \leq 1/m < 5/2$) in the special cases, $\zeta = \ell/10$, $0 \leq \ell \leq 10$, $\ell$ integer. Configurations where the mass point lies on the surface of the sphere are marked by a horizontal dotted line ($1/m = 1/\nu_\Omega = 5/3$). By comparison, the trend related to a classical criterion for the definition of tidal radius [30] is also shown in Fig. 2, where the domain is $0 \leq 1/y_C^* < \zeta/(1 + \zeta)$ and $1/m \to +\infty$ as $1/y_C^* \to \zeta/(1 + \zeta)$. For further details, an interested reader is addressed to Appendix E.

As a guidance example, a sample of 16 globular clusters discussed in an earlier investigation [2] shall be considered, with the addition of Pal5 [10] [23] and the Galactic nuclear star cluster [13] [17]. The position of an assigned globular cluster on the $(O/1$ $y/1$ $m)$ plane can be inferred from the reduced radius, $a_C/a_G$, and the reduced mass, $M_C/M_G$; and the predicted tidal radius is related to the fractional Galactocentric distance, $y_0 = R_0/a_C$, or the parameter, $\zeta = y_0/(y - 1) = R_0/(a_G - a_C)$.

Cluster radii, $a_C$, masses, $M_C$, Galactocentric distances, $R_0$, taken from the above quoted references, are listed in Table 1 with the addition of the inferred $y_0$. With regard to von Hoerner’s criterion, an assigned cluster is expected to show tidal effects according if $1/y > 1/y_C^*$ or $a_C/a_G > a_C^*/a_G$, with the exception of the nuclear stellar cluster (NSC), where the gravitational force from the cluster acts in the same sense as the gravitational force from the Galaxy.

For assigned cluster parameters, the position on the $(O/1$ $y/1$ $m)$ plane, or its logarithmic counterpart, $[O \log(1/y) \log(1/m)]$, depends on the Galaxy mass, $M_G$, and radius, $a_G$. More specifically, increasing/decreasing $M_G$ makes an assigned point shift downwards/upwards and increasing/decreasing $a_G$ makes an assigned point shift leftwards/rightwards. As an exercise, the following values have been considered: $(M_G/10^{10} m_\odot, a_G/kpc) = (5, 25), (5, 125), (50, 125)$, hereafter quoted as case a, b, c, respectively.

The location of globular clusters listed in Table 1 on the $[O \log(1/y) \log(1/m)]$ plane for cases a-c is shown as crosses in Fig. 3, panels a-c, respectively. The whole set of locations is shown in panel d as triangles, diamonds, squares, corresponding to crosses on panels a, b, c, respectively. Configurations related
Table 1: Parameters of globular clusters studied in an earlier paper [2], with the addition of Pal5 [23] for different inferred masses [10] and the Galactic nuclear star cluster, NSC [13] [17]. Column caption: 1 - name (NGC or Pal or NSC); 2 - subsystem (A - [Fe/H] > −1, thick disk; B - old halo; C - young halo); 3 - observed radius, $a_C$/pc; 4 - galactocentric distance, $R_0$/kpc; 5 - decimal logarithm of mass, $\log(M_C/m_{⊙})$; 6 - fractional Galactocentric distance, $y_0 = R_0/a_C$.

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<td>6.7</td>
<td>4.61</td>
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<tr>
<td>6934</td>
<td>C</td>
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<td>14.3</td>
<td>5.39</td>
<td>381</td>
</tr>
<tr>
<td>7078</td>
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<td>65.7</td>
<td>10.4</td>
<td>6.05</td>
<td>158</td>
</tr>
<tr>
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<td>10.4</td>
<td>6.00</td>
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<td>3.78</td>
<td>930</td>
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<td>18.6</td>
<td>3.65</td>
<td>930</td>
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<td></td>
<td>20</td>
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<td>2.98</td>
<td>930</td>
</tr>
<tr>
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<td>0</td>
<td>6</td>
<td>0</td>
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</table>
Potential energies and potential-energy tensors for subsystems

to tidal radii inferred from a global criterion involving homogeneous spheres, Eq. (91), are shown as squares, and their counterparts inferred from a classical local criterion [30] are shown as diamonds, in both cases concerning panels a-c.

The location of Pal5, related to four different inferred masses, is in the lower part of each panel, $\log(1/m) \sim -7$. The location of NSC is near the straight line of unit slope passing through the origin (dotted), as clearly shown in panel d. The NSC tidal radius, inferred from the global criterion, is marked by the square on the extreme right in panels a-c, while the local criterion cannot be applied as $y_0 = R_0/a_C = 0$. NSC values listed in Table 1 [17] are lower with respect to a subsequent investigation [13] by a factor of about 10, which makes related points on the $[O \log(1/y) \log(1/m)]$ plane shift upwards and rightwards along the straight line of unit slope.

With regard to the local criterion, the Galaxy is modelled as a mass point [30] as shown in Appendix E.3, and the results are independent of the Galaxy radius, $a_G$. Accordingly, the distance between crosses and diamonds placed on a horizontal line in panels a, b, (where $a_G$ attains different values) remains unchanged i.e. related points are rigidly shifted. The contrary holds for panel c (where $M_G$ attains a different value). More specifically, cluster radius does not exceed related tidal radius in panels a, b, while the contrary holds in panel c for NGC 5904 and Pal5 (lowest inferred mass) and cluster radius is slightly lower than related tidal radius for NGC 5272, NGC 5466, NGC 6254, and Pal5 (intermediate inferred mass).

The last case (panel c) seems to be more realistic in that (i) the mass contribution from the nonbaryonic dark halo is included, and (ii) NGC 5466 and NGC 5904 show tidal effects [16] [19] with the addition of Pal5 even if, in this case, due to tidal shocks during disk passages [12] [23].

With regard to the global criterion, the Galaxy and the cluster are modelled as homogeneous spheres as shown in Appendix B, and the results depend on both the Galaxy radius, $a_G$, and mass, $M_G$. In particular, $y_0 \sim 100$ from Table 1, which implies $1/m \approx (5/3)\zeta^2(1/y_C^*)$ via Eq. (91), where the remaining terms on the right-hand side can be neglected to a good extent, leaving aside NSC for which $y_0 = 0$, $\zeta = 0$, and $1/m = (1/y_C^*)^3$.

The assumption of homogeneous, spherical-symmetric matter distribution for the Galaxy implies tidal effects are strongly dependent on the Galactocentric distance. For assigned $M_G$ (cases a, b), the cluster radius exceeds related tidal radius provided $R_0 > 4$ kpc in case a and $R_0 > 14$ kpc in case b. For assigned $a_G$ (cases b, c), the cluster radius exceeds related tidal radius provided $R_0 > 14$ kpc in case b and $R_0 > 8$ kpc, $M_C > 10^5 m_\odot$, in case c. Leaving aside NSC, $a_C < a_0^*$ holds for $n = 0$ sample clusters in case a; $n = 12$ in case b; $n = 2$ in case c.

If the Galaxy and globular clusters are modelled as truncated, singular isothermal spheres, then squares on panels a-c of Fig. 3 would place along the
straight line of unit slope passing through the origin (dotted) via Eq. (93), with the exception of NSC for which \( y_0 = 0 \) and Eq. (93) does not hold. Accordingly, \( a_C > a_C^0 \) in any case.

If the Galaxy is modelled as a Roche system and globular clusters as homogeneous spheres, then squares on panels a-c of Fig. 3 would place above the straight line of unit slope passing through the origin (dotted) via Eqs. (94)-(95). Accordingly, \( a_C > a_C^0 \) \textit{a fortiori} in any case.

The global criterion, formulated in Section 4 and used in the current application, cannot predict the occurrence of tidal effects such as the presence of streams and tails. On the other hand, it could provide useful indications on the binding energy of globular clusters within the Galaxy provided more realistic density profiles are considered. In this view, “bound” globular clusters would survive (conceptual) sudden disappearance of the Galaxy, while “unbound” globular clusters would not.

6 Conclusion

Galaxies and galaxy clusters are predicted (via cosmological simulation) or inferred (via data collection) to be made of at least two subsystems (dark nonbaryonic and visible baryonic including leptons) which link only through gravitation, where each component is distorted by tides from the other. In addition, galaxies exhibit bulge-halo and/or bulge-disk structure, where tidal effects even in absence of accretion or merging are a common feature, in that isolated galaxies are an exception rather than a rule.

With these ideas in mind and restricting to two-component systems, attention has been focused on general properties of potential energies and potential-energy tensors, including related first variations and physical interpretation. In addition, a global criterion for the definition of tidal radius has been proposed and a few guidance examples, restricted to special density profiles, have been shown.

An application has been made to a sample of globular clusters within the Galaxy, considered in an earlier investigation [2], with the addition of Pal5 [23] for different inferred masses [10] and the Galactic nuclear star cluster [13] [17]. In particular, the extent to which the above mentioned global criterion could provide useful indications on the binding energy of globular clusters, has been analysed by comparison with the results from a classical local criterion [30].

The main results of the current paper may be summarized as follows.

(1) An explicit expression has been determined for the first variations of subsystem potential energies and potential-energy tensors, which could be useful for e.g., practical use of virial equations in linearized form for the treatment of the stability of a configuration [11] Chap. 3, §23, the effect
of viscous dissipation on the stability [11] Chap. 5, §37; Chap. 8, §59, the
determination of bifurcation points [11] Chap. 6, §45 and loci of neutral

(2) A physical interpretation has been proposed for the potential interaction
and potential tidal energy, in addition to the well known interpretation
of the potential self energy e.g., [22] Chap. III, §76. More specifically, the
potential interaction energy, \( W_{uv} \), represents the amount of work which
must be done upon the subsystem, \( u \), as a whole, in order to recede up to
an infinite distance from the subsystem, \( v \), preserving virial equilibrium.
On the other hand, the potential tidal energy, \( V_{uv} \), represents the change
(regardless of the sign) in potential self energy, \( \Omega_u \), that is necessary for
u subsystem maintains virial equilibrium in absence of v subsystem.

(3) A global criterion for the definition of subsystem tidal radius has been
inferred by requiring null total energy for \( u \) subsystem in absence of \( v \)
subsystem, which implies the potential self energy equals the potential
tidal energy, \( \Omega_u = V_{uv} \), regardless of density profile and slope.

(4) Restricting to spherical-symmetric mass distributions, one completely lying
within the other, the dependence of the reduced mass, \( 1/m = M_i/M_j \),
on the reduced tidal radius of the embedded sphere, \( 1/y^*_i = a^*_i/a_j \), for
assigned fractional distance between the centre of the embedded and the
embedding sphere, \( y_0 = \zeta(y - 1) \), has been determined for a few special density profiles, namely (a) both homogeneous; (b) both truncated, singular isothermal; (c) homogeneous and Roche system i.e. mass point surrounded by a vanishing atmosphere. Related trends have been compared with their counterparts inferred from a classical local criterion for the definition of subsystem tidal radius [30].

(5) An application has been made to a sample of Galactic globular clusters
with the addition of the Galactic nuclear star cluster (NSC), for different values of Galaxy mass and radius. In the more realistic case, 
\( (M_G/10^{10}m_\odot, a_G/kpc) = (50, 125) \), \( a_C > a^*_C \), according to the local criterion, for two sample clusters which also show tidal effects, and \( a_C < a^*_C \),
according to the global criterion, for two sample clusters and NSC.

References


Potential energies and potential-energy tensors for subsystems


Further insight on the physical interpretation of the potential tidal energy and the potential interaction energy can be gained via the following conceptual experiment. Let $i, j$, be subsystems in virial equilibrium, under the action of gravitation only. Let $u = i, j$, be a generic subsystem and $v = j, i$, the remaining one. Accordingly, the condition of virial equilibrium and the total energy for $u$ subsystem are expressed by Eqs. (55) and (56), respectively. Let the following processes take place with regard to $u$ subsystem.

First, particles are instantaneously halted and their kinetic energy is converted into potential energy, via infinitely compressible and perfectly elastic springs say, one per particle, which implies particles are at rest with respect to the cosmic background radiation, say. Second, particles are instantaneously
Table 2: Sequential states of $u$ subsystem during the conceptual experiment discussed in the text, related transition time ($\Delta t$), potential self energy (PSE), potential tidal energy (PTE), potential interaction energy (PIE), kinetic energy (KE), and particle status. The initial state, by definition, relates to a null transition time. Sequential states of $v$ subsystem can be inferred by replacing the index, $u$, with the index, $v$, and vice versa.

<table>
<thead>
<tr>
<th>state</th>
<th>$\Delta t$</th>
<th>PSE</th>
<th>PTE</th>
<th>PIE</th>
<th>KE</th>
<th>status</th>
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</thead>
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<tr>
<td>1</td>
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<td>$\Omega_u$</td>
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<td>$W_{uv}$</td>
<td>$T_u$</td>
<td>virialized</td>
</tr>
<tr>
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<td>0</td>
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<td>$V_{uv}$</td>
<td>$W_{uv}$</td>
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<td>halted</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>$\Omega_u$</td>
<td>$V_{uv}$</td>
<td>$W_{uv}$</td>
<td>0</td>
<td>connected</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>$\Omega_u$</td>
<td>$V_{uv}$</td>
<td>$W_{uv}$</td>
<td>$T_u$</td>
<td>started</td>
</tr>
<tr>
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<td>$\infty$</td>
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<td>0</td>
<td>$T_u$</td>
<td>translated</td>
</tr>
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<td>6</td>
<td>0</td>
<td>$\Omega_u$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>halted</td>
</tr>
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<td>0</td>
<td>0</td>
<td>$T'_u$</td>
<td>virialized</td>
</tr>
</tbody>
</table>

connected, one with the remaining others, via rigid, massless, undeformable, infinitely thin rods to ensure potential self energy conservation. Third, the potential energy stored into compressed springs is converted into translation kinetic energy of the centre of mass, which makes the subsystem, $u$, move rigidly along a straight line at velocity, $(2T_u/M_u)^{1/2}$. Fourth, the centre of mass is instantaneously halted by storing again the kinetic energy, $T_u$, into compressed springs as before starting the translation. Fifth, particles are instantaneously disconnected. Sixth, particles are instantaneously restored free via conversion of potential energy within springs into kinetic energy, keeping the centre of mass at rest. Seventh, particles are relaxed owing to the absence of $v$ subsystem. Eighth, particle are virialized attaining a new equilibrium configuration.

The above mentioned states are summarized in Table 2, where the following quantities are listed: the transition time ($\Delta t$), the potential self energy (PSE), the potential tidal energy (PTE), the potential interaction energy (PIE), the kinetic energy (KE), and the particle status. All the transitions are conceived as instantaneous ($\Delta t = 0$) that is true, by definition, for the initial state, with the exception of the translation to infinite distance, which needs an infinite time, and the last virialization, which is completed in a relaxation time, $\tau$.

The kinetic energy is null in the state 2, 3, 6, 7, but an equivalent amount is stored as potential energy into compressed springs, according to the above considerations. The kinetic energy is partly due to systematic motions and
partly to random motions in the state 1, entirely due to systematic motions in the state 4, 5, partly due to systematic motions and partly due to random motions in the state 8, 9.

In particular, the orbital kinetic energy in the state 1 cannot be converted into translation kinetic energy in the state 8 to ensure subsystem confinement within a limited region of space, implying virial equilibrium e.g., [5] Chap. II, §10. More specifically, the orbital kinetic energy can be preserved in macroscopic form via conversion into systematic rotation, or initially preserved in macroscopic form via conversion into radial oscillations and progressively turned into microscopic form via violent relaxation [21].

The transition 1-4 violates the second principle of thermodynamics in that kinetic energy is transferred from random motions to translation motions via counterparts of Maxwell’s daemons, who are able to halt (via infinitely compressible and perfectly elastic springs) and connect (via rigid, massless, undeformable, infinitely thin rods) particles. On the other hand, the total energy is left unchanged and Eqs. (55)-(56) hold.

The reverse occurs for the transition 5-8, where daemons act to transfer kinetic energy from translation motions to random motions, conformly to the second principle of thermodynamics, leaving the total energy unchanged even if intrinsically different with respect to the transition 1-4. Accordingly, \( E''_u = \Omega_u + T_u \) via Eqs. (56) and (60). The subsystem, \( u \), virializes through the transition 8-9, and Eq. (61) holds. The transition 1-8 is equivalent to the instantaneous disappearence of \( v \) subsystem keeping the centre of mass of \( u \) subsystem at rest and leaving the kinetic energy, \( T_u \), unchanged, as assumed in Section 3.

### B Homogeneous spheres one completely lying within the other

Let \((O_i x_1 x_2 x_3)\) and \((O_j X_1 X_2 X_3)\) be Cartesian reference frames with origin placed on the centre of the embedded and embedding sphere, respectively, coinciding axes, \(x_1, X_1\), and parallel axes, \(x_2, X_2; x_3, X_3\). Let \(P(x_1, x_2, x_3) = P(X_1, X_2, X_3)\) be a generic point of the embedded sphere, \(r = (x_1^2 + x_2^2 + x_3^2)^{1/2}\) the radial coordinate of \(P\) with respect to \((O_i x_1 x_2 x_3)\), and \(R = (X_1^2 + X_2^2 + X_3^2)^{1/2}\), \(R_0 = (X_{01}^2 + X_{02}^2 + X_{03}^2)^{1/2}\) the radial coordinates of \(P, O_i\), with respect to \((O_j X_1 X_2 X_3)\).

Accordingly, Cartesian coordinates are related as:

\[
X_s = x_s + \delta_{1s}R_0 \quad ; \quad s = 1, 2, 3 \quad ;
\]

where \(\delta_{pq}\) is the Kronecker symbol, and radial coordinates are related as:

\[
R^2 = R_0^2 + 2R_0x_1 + r^2 \quad ;
\]

where \(\delta_{pq}\) is the Kronecker symbol, and radial coordinates are related as:
The potential-energy tensors, \((W_{ij})_{pq}\) and \((V_{ij})_{pq}\), can be determined from the substitution of Eq. (102), (104), into (15), (16), respectively, and related integration on the volume of the embedded sphere, \(S_i = (4\pi/3)a_1^3\). In spherical coordinates, the infinitesimal volume element reads:

\[
d^3S_i = dx_1 \, dx_2 \, dx_3 = r^2 \sin \theta \, dr \, d\theta \, d\phi ;
\]

\[
x_1 = r \sin \theta \cos \phi ; \quad x_2 = r \sin \theta \sin \phi ; \quad x_3 = r \cos \theta ;
\]

\[
0 \leq \phi \leq 2\pi ; \quad 0 \leq \theta \leq \pi ; \quad 0 \leq r \leq a_1 ;
\]
and the following relations hold:

\[
\int_{S_1} x_p x_q \, d^3 S = \delta_{pq} \frac{1}{5} S_i a_i^2 \quad ; \quad p = 1, 2, 3 \quad ; \quad q = 1, 2, 3 \quad ;
\]

(109)

\[
\int_{S_1} x_s \, d^3 S = 0 \quad ; \quad s = 1, 2, 3 \quad ;
\]

(110)

after transformation of Cartesian into spherical coordinates e.g., [28] Chap. 22, §§22.81-83.

Accordingly, the integration of the right-hand side of Eqs. (15) and (16) via (102)-(110) yields:

\[
(W_{ij})_{pq} = -\frac{1}{5} \delta_{pq} \frac{GM^2_i m}{a_i y^3} \left[ \frac{5}{4} y^2 - \frac{1}{4} + \left( \delta_{1p} - \frac{3}{4} \right) y_0^2 \right] \quad ; \quad (W_{ji})_{pq} = (W_{ij})_{pq} ;
\]

(111)

\[
(V_{ij})_{pq} = -\frac{1}{5} \delta_{pq} \frac{GM^2_i m}{a_i y^3} \left( 1 + 5 \delta_{1p} y_0^2 \right) ;
\]

(112)

and Eq. (37) takes the explicit form:

\[
(Q_{ij})_{pq} = -\frac{1}{5} \delta_{pq} \frac{GM^2_i m}{a_i y^3} \left[ \frac{5}{4} (1 - y^2) + \left( 4 \delta_{1p} + \frac{3}{4} \right) y_0^2 \right] ;
\]

\[
(Q_{ji})_{pq} = -(Q_{ij})_{pq} ;
\]

(113)

finally, using Eqs. (37), (39), (40), yields:

\[
(V_{ji})_{pq} = (W_{ji})_{pq} + (Q_{ji})_{pq} = (W_{ij})_{pq} - (Q_{ij})_{pq} ;
\]

(114)

and the substitution of Eqs. (111) and (113) into (114) produces:

\[
(V_{ji})_{pq} = -\frac{1}{5} \delta_{pq} \frac{GM^2_i m}{a_i y^3} \left[ \frac{5}{2} y^2 - \frac{3}{2} + \frac{3}{2} (2 \delta_{1p} + 1) y_0^2 \right] ;
\]

(115)

which completes the determination of potential-energy tensors in the case under discussion. Related traces are expressed by Eqs. (77)-(79).

C Reduced mass vs. reduced virial radius of the embedding sphere

With regard to the embedding and the embedded sphere, the reduced radius, \(y = a_j/a_i\), by definition satisfies the inequality, \(y \geq 1\). Then the fraction on the right-hand side of Eq. (89), \(y_0^s = N/D\), satisfies either \(N \geq D > 0\) or \(N \leq D < 0\). An additional condition, expressed by Eq. (88), implying real solutions of the second-degree equation, Eq. (85), is a nonnegative discriminant, \(\Delta \geq 0\).
The former alternative, $N \geq D > 0$, implies the following relation:

$$\mp [5(3 + 2\zeta^2) - 2m(3 + 5\zeta^2)]^{1/2} \geq 5 - 2m ; \quad 0 \leq \zeta \leq 1 ; \quad (116)$$

where $5 - 2m \geq 5(1 - \zeta^2) - 2m = D > 0$, which rules out the minus on the left-hand side. Accordingly, Eq. (116) is equivalent to:

$$5(3 + 2\zeta^2) - 2m(3 + 5\zeta^2) \geq (5 - 2m)^2 ; \quad (117)$$

which can be ordered in $m$ as:

$$2m^2 - (7 - 5\zeta^2)m + 5(1 - \zeta^2) \leq 0 ; \quad (118)$$

where the solutions of the associated equation are:

$$m_1 = 1 ; \quad m_2 = \frac{5}{2}(1 - \zeta^2) ; \quad (119)$$

and the solution of the disequation reads:

$$\min \left[1, \frac{5}{2}(1 - \zeta^2)\right] \leq m \leq \max \left[1, \frac{5}{2}(1 - \zeta^2)\right] ; \quad (120)$$

on the other hand the condition, $D > 0$, is equivalent to:

$$m < \frac{5}{2}(1 - \zeta^2) ; \quad (121)$$

and the combination of Eqs. (120) and (121) yields:

$$1 \leq m < \frac{5}{2}(1 - \zeta^2) ; \quad 0 \leq \zeta \leq \sqrt{\frac{3}{5}} ; \quad (122)$$

which is the domain of reduced mass, $m = M_j/M_i$, related to the reduced tidal radius, $y_j^* = a_j^*/a_i$, in the case under discussion.

The latter alternative, $N \leq D < 0$, implies the following relation:

$$\mp [5(3 + 2\zeta^2) - 2m(3 + 5\zeta^2)]^{1/2} \leq 5 - 2m ; \quad 0 \leq \zeta \leq 1 ; \quad (123)$$

where $5 - 2m > 0$, owing to Eq. (88). Accordingly, the minus on the left-hand side of Eq. (123) can be erased in that the remaining inequality implies the validity of both.

Following a similar procedure as in the former case, the solution of the disequation reads:

$$m \leq \min \left[1, \frac{5}{2}(1 - \zeta^2)\right] ; \quad m \geq \max \left[1, \frac{5}{2}(1 - \zeta^2)\right] ; \quad (124)$$
on the other hand the condition, $D < 0$, is equivalent to:

$$m > \frac{5}{2}(1 - \zeta^2) ;$$  \hspace{1cm} (125)

and the combination of Eqs. (124), (125), and (88) yields:

$$\max \left[1, \frac{5}{2}(1 - \zeta^2) \right] \leq m \leq \frac{5}{2} \left( \frac{3 + 2\zeta^2}{3 + 5\zeta^2} \right) < \frac{5}{2} ; \hspace{0.5cm} 0 < \zeta \leq 1 ;$$  \hspace{1cm} (126)

which is the domain of reduced mass, $m = M_j/M_i$, related to the reduced tidal radius, $y_j^* = a_j^*/a_i$, in the case under discussion.

The combination of Eqs. (122) and (126) yields Eq. (90).

\section{Truncated, singular isothermal spheres, one completely lying within the other}

Potential energies and potential-energy tensors of truncated, singular isothermal spheres, one completely lying within the other, can be expressed in simple form only if the centre of the embedded sphere is sufficiently distant from the centre of the embedding sphere. The result is [10]:

$$\Omega_i = -\frac{GM_i^2}{a_i} ; \hspace{1cm} \Omega_j = -\frac{GM_j^2}{a_j} ;$$  \hspace{1cm} (127)

$$W_{ij} = -\frac{1}{2} \frac{GM_i^2}{a_i} m \left[ 1 - \ln \frac{y_0}{y} - \frac{1}{18} \frac{1}{y_0^2} \right] ; \hspace{0.5cm} y > y_0 \gg 1 ;$$  \hspace{1cm} (128)

$$V_{ij} = W_{ij} + Q_{ij} = -\frac{GM_i^2}{a_i} m \left[ 1 + \ln \frac{y_0}{y} + \frac{1}{18} \frac{1}{y_0^2} \right] ; \hspace{0.5cm} y > y_0 \gg 1 ;$$  \hspace{1cm} (129)

$$Q_{ij} = -\frac{1}{2} \frac{GM_i^2}{a_i} m \left[ 1 + \ln \frac{y_0}{y} + \frac{1}{18} \frac{1}{y_0^2} \right] ; \hspace{0.5cm} y > y_0 \gg 1 ;$$  \hspace{1cm} (130)

$$W_{ji} = W_{ij} ; \hspace{1cm} Q_{ji} = -Q_{ij} ;$$  \hspace{1cm} (131)

$$V_{ji} = W_{ji} + Q_{ji} = \frac{GM_i^2}{a_i} m \left[ \ln \frac{y_0}{y} + \frac{1}{18} \frac{1}{y_0^2} \right] ; \hspace{0.5cm} y > y_0 \gg 1 ;$$  \hspace{1cm} (132)

accordingly, Eq. (75) takes the explicit form:

$$\frac{m}{y} = 1 ; \hspace{1cm} y > y_0 \gg 1 ;$$  \hspace{1cm} (133)

$$\frac{m}{y} \left[ \ln \frac{y_0}{y} + \frac{1}{18} \frac{1}{y_0^2} \right] = -\frac{m^2}{y} ; \hspace{0.5cm} y > y_0 \gg 1 ;$$  \hspace{1cm} (134)

for $i$ and $j$ subsystem, respectively.
The reduced mass, \(1/m = M_i/M_j\) and \(m = M_j/M_i\), in terms of the reduced tidal radius, \(1/y_i^* = a_i^*/a_i\) and \(y_j^* = a_j^*/a_i\), can be inferred from Eqs. (133) and (134), respectively, via (83). The result is:

\[
\frac{1}{m} = \frac{1}{y_i^*} ; \quad 0 \ll \zeta \leq 1 ; \quad (135)
\]

\[
m = -\ln \zeta - \ln \left(1 - \frac{1}{y_j^*}\right) - \frac{1}{18} \frac{1}{\zeta^2} \left(\frac{1}{y_j^*}\right)^2 \left(1 - \frac{1}{y_j^*}\right)^2 ; \quad 0 \ll \zeta \leq 1 ; \quad (136)
\]

where \(m \to -\infty\) as \(y_j^* \to 1^+\), \(m \to -\ln \zeta \to +\infty\), and the absence of extremum points implies the existence of a single zero, \(y_{0,j}^*\), for \(m\). Accordingly, the tidal radius of the embedding sphere can be defined within the range, \(y_j^* > y_{0,j}^*\), keeping in mind \(y_{0,j}^* \to +\infty\) as \(\zeta \to 1^-\).

In the special case of a globular cluster (\(i = C\)) within the Galaxy (\(j = G\), Eq. (135) reduces to (93).

E. A heterogeneous sphere completely lying within a Roche system

A Roche system (mass point surrounded by a vanishing atmosphere) is preferred to a “naked” mass point in that it can be conceived as a heterogeneous sphere with infinite concentration. Let \((OX_1 X_2 X_3)\) be a Cartesian reference frame with the origin placed on the mass point and axis, \(X_1\), passing through the centre of the heterogeneous sphere. Let \(R_0 = (X_{01}^2 + X_{02}^2 + X_{03}^2)^{1/2} = X_{01}\) be the distance between the centre and the mass point. As the tidal radius cannot be defined for a mass point, considerations shall be restricted to the heterogeneous sphere. Let \(\overline{R}_0\) be the radius of a fictitious circular orbit of the centre of the sphere around the mass point where the virial theorem is satisfied and, in consequence, related potential and kinetic energy equal the mean values along the real orbit. Further attention shall be restricted to the fictitious orbit for simplicity, hence \(\overline{R}_0 = R_0\). In general, the mass point can be inside or outside the heterogeneous sphere. The two possibilities shall be discussed separately.

E.1 Mass point outside the heterogeneous sphere

The gravitational potential of the heterogeneous sphere in \(O\) is e.g., [22] Chap. II, §29:

\[
\Psi_i(O) = \frac{GM_i}{R_0} ; \quad (137)
\]
and, in addition:

\[
\left( \frac{\partial V_i}{\partial x_s} \right)_O = \left( \frac{\partial V_i}{\partial R} \frac{\partial R}{\partial x_s} \right)_O = -\frac{G M_i X_0 s}{R_0^2} = -\delta_{1s} \frac{G M_i}{R_0^2} ;
\]

(138)

\[
\sum_{s=1}^{3} \left( \frac{\partial V_i}{\partial x_s} X_s \right)_O = 0 ;
\]

(139)

where \( \delta_{pq} \) is the Kronecker symbol.

Related potential energies are:

\[
\Omega_i = -\nu_\Omega \frac{G M_i^2}{a_i} ;
\]

(140)

\[
W_{ji} = -\frac{1}{2} M_j V_i(O) = -\frac{1}{2} \frac{G M_i M_j}{R_0} ;
\]

(141)

\[
V_{ji} = M_j \sum_{s=1}^{3} \left( \frac{\partial V_i}{\partial x_s} X_s \right)_O = 0 ;
\]

(142)

\[
Q_{ji} = V_{ji} - W_{ji} = \frac{1}{2} \frac{G M_i M_j}{R_0} ;
\]

(143)

\[
V_{ij} = W_{ij} + Q_{ij} = -\frac{G M_i M_j}{R_0} ;
\]

(144)

where the last relation is owing to the symmetry of the potential interaction energy via Eq. (41), \( W_{ij} = W_{ji} \), and to the antisymmetry of the potential residual energy via Eq. (42), \( Q_{ij} = -Q_{ji} \).

With regard to the heterogeneous sphere, Eq. (75) via (80), (83), (140), (144), takes the explicit form:

\[
\nu_\Omega = \frac{1}{\zeta} \frac{m}{y - 1} ; \quad y_0 = \zeta(y - 1) \geq 1 ; \quad y \geq \frac{1 + \zeta}{\zeta} ; \quad 0 \leq \zeta \leq 1 ;
\]

(145)

where \( \nu_\Omega \) is a factor which depends on the density profile within the sphere e.g., \( \nu_\Omega = 3/5 \) for the homogeneous sphere and \( \nu_\Omega = 1 \) for the truncated, singular isothermal sphere.

The reduced mass, \( 1/m = M_i M_j \), in terms of the reduced tidal radius, \( 1/y_i^* = a_i^* / a_j \), can be inferred from Eq. (145) as:

\[
\frac{1}{m} = \frac{1}{\nu_\Omega} \frac{1}{\zeta} \frac{1}{y_i^*} \left( 1 - \frac{1}{y_i^*} \right)^{-1} ; \quad 0 < \frac{1}{y_i^*} \leq \frac{\zeta}{1 + \zeta} ; \quad 0 \leq \zeta \leq 1 ;
\]

(146)

where \( 1/m = 1/\nu_\Omega \) as \( 1/y_i^* = \zeta / (1 + \zeta) \). In the special case of a globular cluster \( (i = C) \) within the Galaxy \( (j = G) \), Eq. (146) reduces to (94).
E.2 Mass point inside the heterogeneous sphere

The gravitational potential of the heterogeneous sphere in $O$ is e.g., [3]:

\[ \mathcal{V}_i(O) = \mathcal{V}_i^{(\text{int})}(O) + \mathcal{V}_i^{(\text{ext})}(O) \rightleftharpoons (147) \]
\[ \mathcal{V}_i^{(\text{ext})}(O) = \frac{GM_i(R_0)}{R_0} = \frac{GM_i M_i(R_0)}{M_i} \frac{a_i}{R_0}; \rightleftharpoons (148) \]
\[ \mathcal{V}_i^{(\text{int})}(O) = 2\pi G \rho_0 a_i^2 F(\xi_O); \rightleftharpoons (149) \]
\[ F(\xi) = 2 \int_{\xi}^{1} f(\xi) \xi \, d\xi; \rightleftharpoons (150) \]
\[ \rho_i(r) = \rho_0 f(\xi) \rightleftharpoons f(0) = 1 \quad \xi = \frac{r}{a_i}; \rightleftharpoons (151) \]

where $\rho_0$ is the central density, $\rho_i(r)$ the density profile, $M_i(r)$ the mass distribution, $\xi$ a reduced radial coordinate, $\xi_O = R_0/a_i = y_0 \leq 1$.

From this point on, attention shall be restricted to a homogeneous sphere for simplicity, which implies $\rho_0 = M_i/(4\pi a_i^3)$, $M_i(R)/M_i = R^3/a_i^3$; $f(\xi) = 1$, $F(\xi) = 1 - \xi^2$. Accordingly, Eqs. (148) and (149) reduce to:

\[ \mathcal{V}_i^{(\text{ext})}(O) = \frac{GM_i R_0^2}{a_i a_i^2}; \rightleftharpoons (152) \]
\[ \mathcal{V}_i^{(\text{int})}(O) = \frac{3 GM_i}{2 a_i} \left( 1 - \frac{R_0^2}{a_i^2} \right); \rightleftharpoons (153) \]

and Eq. (147) takes the explicit form:

\[ \mathcal{V}_i(O) = \frac{GM_i}{a_i} \left( \frac{3}{2} - \frac{1}{2} \frac{R_0^2}{a_i^2} \right); \rightleftharpoons (154) \]

accordingly, related potential energies are:

\[ W_{ji} = -\frac{1}{2} M_j \mathcal{V}_i(O) = -\frac{1}{2} \frac{GM_i M_j}{a_i} \left( \frac{3}{2} - \frac{1}{2} \frac{R_0^2}{a_i^2} \right); \rightleftharpoons (155) \]
\[ V_{ji} = M_j \sum_{s=1}^{3} \left( \frac{\partial \mathcal{V}_i}{\partial X_s} X_s \right) \bigg|_O = 0; \rightleftharpoons (156) \]
\[ Q_{ji} = V_{ji} - W_{ji} = \frac{1}{2} \frac{GM_i M_j}{a_i} \left( \frac{3}{2} - \frac{1}{2} \frac{R_0^2}{a_i^2} \right); \rightleftharpoons (157) \]
\[ V_{ij} = W_{ij} + Q_{ij} = -\frac{1}{2} \frac{GM_i M_j}{a_i} \left( \frac{3}{2} - \frac{1}{2} \frac{R_0^2}{a_i^2} \right); \rightleftharpoons (158) \]

where the last relation is owing to the symmetry of the potential interaction energy via Eq. (41), $W_{ij} = W_{ji}$, and the antisymmetry of the potential residual energy via Eq. (42), $Q_{ij} = -Q_{ji}$.
With regard to the homogeneous sphere, Eq. (75) via (80), (83), (140), (158), takes the explicit form:
\[
\nu_{\Omega} = m \left[ \frac{3}{2} - \frac{1}{2} \xi^2 (y - 1)^2 \right] ; \quad y_0 = \zeta (y - 1) \leq 1 ; \quad y \leq \frac{1 + \zeta}{\zeta} ; \quad (159)
\]
where \( \nu_{\Omega} = 3/5 \) in the case under discussion.

The reduced mass, \( 1/m = M_i/M_j \), in terms of the reduced tidal radius, \( 1/y_i^* = a_i^*/a_j \), can be inferred from Eq. (159) as:
\[
\frac{1}{m} = \frac{5}{3} \left[ \frac{3}{2} - \frac{1}{2} \xi^2 \left( \frac{1}{y_i^*} \right)^{-2} \right] ; \quad 1 \geq \frac{1}{y_i^*} \geq \frac{\zeta}{1 + \zeta} ; \quad 0 \leq \xi \leq 1 ; \quad (160)
\]
where \( 1/m = 1/\nu_{\Omega} = 5/3 \) as \( 1/y_i^* = \zeta/(1 + \zeta) \), \( 1/m = 5/2 \) as \( 1/y_i^* = 1 \), and \( y_i^* \geq 1 \) by definition.

In the special case of a globular cluster \( (i = C) \) within the Galaxy \( (j = G) \), Eq. (160) reduces to (95).

### E.3 von Hoerner’s tidal radius

Let a heterogeneous sphere, centered on \( O_i \), be subjected to the gravitational force from a mass point placed on \( O_j \) outside the boundary. Let \( a_i \) be the radius of the sphere and \( R_0 \) the distance \( O_j O_i \), where \( R_0 > a_i \) in the case under consideration. Let \( P \) be the intersection point between the boundary and the segment, \( O_j O_i \).

The gravitational force from the mass point on a unit mass placed on \( O_i \) and \( P \), respectively, is:
\[
F_{G,j}(O_i) = -\frac{GM_j}{R_0^2} ; \quad F_{G,j}(P) = -\frac{GM_j}{(R_0 - a_i)^2} ; \quad (161)
\]
and the gravitational force from the heterogeneous sphere on a unit mass placed on \( P \) is:
\[
F_{G,i}(P) = \frac{GM_i}{a_i^2} ; \quad (162)
\]
which has an opposite orientation with respect to \( F_{G,j} \).

Accordingly, a local criterion for the definition of tidal radius reads:
\[-F_{G,j}(O_i) + F_{G,j}(P) + F_{G,i}(P) = 0 \]
which, by use of Eqs. (161)-(162), after little algebra takes the explicit expression:
\[
\frac{a_i^2}{R_0^2} \frac{2R_0 a_i - a_i^2}{(R_0 - a_i)^2} = \frac{M_i}{M_j} ; \quad (163)
\]
where the limit, \( R_0 \gg a_i \), yields the classical result [30].

The reduced mass, \( 1/m = M_i/M_j \), in terms of the reduced tidal radius, \( 1/y_i^* = a_i^*/a_j \), can be inferred from Eq. (163) via (80), (83), as:

\[
\frac{1}{m} = \frac{1}{\zeta^3} \left[ \frac{1}{y_i^*} \left( 1 - \frac{1}{y_i^*} \right)^{-1} \right]^3 \left[ 2 - \frac{1}{\zeta y_i^*} \left( 1 - \frac{1}{y_i^*} \right)^{-1} \right] \\
\times \left[ 1 - \frac{1}{\zeta y_i^*} \left( 1 - \frac{1}{y_i^*} \right)^{-1} \right]^{-2} ; \quad y_i^* > 1 ; \quad 0 \leq \zeta \leq 1 ; \quad (164)
\]

where \( 1/m \to -\infty \) as \( 1/y_i^* \to 1^- \), \( 1/m \to 0 \) as \( 1/y_i^* \to 0 \), and \( 1/m = 0 \) as \( 1/y_i^* = 2\zeta/(1 + 2\zeta) \). Keeping in mind \( m \geq 0 \) by definition, the domain of the function, expressed by Eq. (164) for assigned \( \zeta \), reads:

\[
0 < \frac{1}{y_i^*} \leq \frac{2\zeta}{1 + 2\zeta} ; \quad 0 \leq \zeta \leq 1 ; \quad (165)
\]

while, on the other hand, the condition that the mass point lies outside the sphere implies \( y_0 > 1 \) or:

\[
\frac{1}{y_i^*} < \frac{\zeta}{1 + \zeta} \leq \frac{2\zeta}{1 + 2\zeta} ; \quad (166)
\]

where \( 1/m \to +\infty \) as \( 1/y_i^* \to \zeta/(1 + \zeta) \), and the domain under discussion reduces to:

\[
0 < \frac{1}{y_i^*} < \frac{\zeta}{1 + \zeta} ; \quad 0 \leq \zeta \leq 1 ; \quad (167)
\]

conformly to Eqs. (165) and (166).

The comparison of Eq. (164), inferred using a classical local criterion for the definition of tidal radius [30] with its counterpart inferred using a global criterion, Eq. (146) for a homogeneous sphere, discloses two main differences, namely (i) the tidal radius is independent of the density profile in the former case but the contrary holds in the latter and, (ii) the reduced mass, \( 1/m \), exhibits a cubic dependence on the reduced tidal radius, \( 1/y_i^* \), in the former case and a linear dependence in the latter, provided \( y_i^* \ll 1 \).
Figure 2: The reduced mass, $1/m = M_C/M_G$, vs. the reduced tidal radius, $1/y_C^* = a_C^*/a_G$, for different models of globular cluster within the Galaxy: (i) homogeneous spheres one completely lying within the other [from (0,0) to (1,1)]; (ii) truncated, singular isothermal spheres one completely lying within the other (dashed line); (iii) a homogeneous sphere completely lying within a Roche system [from (0,0) to (1.5/2); the locus of configurations where the mass point lies on the boundary of the sphere, $1/m = 5/3$, is represented by the dotted horizontal line]. Divergent curves are determined according to (iv) von Hoerner’s criterion for the definition of tidal radius. Different lines of the same kind relate to $\zeta = \ell/10$, $0 \leq \ell \leq 10$, $\ell$ integer, from bottom to top [model (i)]; from left to right [models (iii) and (iv)], where the vertical axis and the origin, respectively, correspond to $\zeta = 0$; with no correlation, implying coincident lines [model (ii)]. See text for further details.
Figure 3: Location of globular clusters listed in Table 1 on the \( \log(1/y) \log(1/m) \) plane, for assumed Galaxy mass and radius, \((M_G/10^{10} m_\odot, a_G/kpc)\), equal to (5, 25), (5, 125), (50, 125), panels a, b, c, respectively (crosses). The whole set of locations is collected in panel d as triangles, diamonds, squares, corresponding to crosses on panels a, b, c, respectively. Configurations related to tidal radii inferred from a global criterion involving homogeneous spheres are shown as squares, and their counterparts inferred from a classical local criterion [30] are shown as diamonds, in both cases concerning panels a-c. The location of Pal15, related to four different inferred masses, is in the lower part of each panel, \( \log(1/m) \lesssim -7 \). The location of the nuclear star cluster (NSC) is near the the straight line of unit slope passing through the origin (dotted), as can be seen in panel d. The NSC tidal radius inferred from the global criterion is marked by the last square on the right in panels a-c, while the local criterion cannot be applied in this case. See text for further details. Warning: symbol caption in panels a-c is different with respect to panel d, as explained above.