Computation of Eigenvalues
of Discrete Lower Semibounded Operators

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Abstract

Based on the Galerkin method, the formulas for eigenvalues of discrete lower semibounded operators were obtained. Numerical experiments demonstrated high accuracy and computational efficiency of them.

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1 Introduction

Based on the method of regularized traces a noniterative method was developed [1] – [6]. This method enables us to compute eigenvalues of perturbed discrete lower semibounded operators.

Take a discrete lower semibounded operator $T$ and a bounded operator $P$ on a separable Hilbert space $H$. Assume that the eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ and
orthonormal eigenfunctions \( \{v_n\}_{n=1}^{\infty} \) of \( T \) are available and enumerate them in the ascending order of \( \lambda_n \), taking their multiplicities \( \nu_n \) into account. Denote by \( n_0 \) the number of distinct eigenvalues \( \lambda_n \) lying inside the circle \( T_{n_0} \) of radius \( \rho_{n_0} = \frac{|\lambda_{n_0} + 1 + \lambda_{n_0}|}{2} \) centered at the origin. Enumerate as \( \{\mu_n\}_{n=1}^{\infty} \) the eigenvalues of the operator \( T + P \) in the ascending order of their real parts, taking into account their algebraic multiplicities.

If \( q_n = \frac{2\|P\|}{|\lambda_n + e_n - \lambda_n|} < 1 \) for all \( n \in \mathbb{N} \) and the system of eigenfunctions \( \{v_n\}_{n=1}^{\infty} \) of \( T \) constitutes an orthonormal basis of \( H \) then \( m_0 = \sum_{n=1}^{n_0} \nu_n \) and we can calculate the eigenvalues \( \{\mu_n\}_{n=1}^{m_0} \) of \( T + P \) as

\[
\mu_n = \lambda_n + (Pv_n, v_n) + \tilde{\delta}(n), \quad n = 1, m_0,
\]

where \( \tilde{\delta}(n) = \delta(n) - \delta(n-1) \) with \( \delta(n) = \sum_{k=1}^{n} [\mu_k - \tilde{\mu}_k(n)] \) and \( \tilde{\mu}_k(n) \) is the \( n \)-th Galerkin approximation to the corresponding eigenvalue \( \mu_k \) of \( T + P \). For \( \tilde{\delta}(n) \) we have the estimates

\[
|\tilde{\delta}(n)| \leq (2n - 1) \rho_n \frac{q^2}{1 - q}, \quad q = \max_{n \in \mathbb{N}} q_n
\]

The method of regularized traces was the basis of many successful studies, including studies of eigenvalue problem in the stochastic models\[7\]. Unfortunately, restrictions on the norm of the perturbing operator limits the usefulness of this method and as a consequence formulas (2). Further investigations showed that we can remove the restrictions on the norm if we use the Galerkin method.

### 2 The Formulas of the Eigenvalues of Discrete Lower Semibounded Operators

Consider a discrete lower semibounded operator \( L \) on a separable Hilbert space \( H \). We determine the eigenvalues \( \mu \) when finding the solutions of the operator equation

\[
Lu = \mu u
\]

satisfying certain homogeneous boundary conditions.

To find the eigenvalues of \( L \), we use the Galerkin method. Introduce a sequence \( \{H_n\}_{n=1}^{\infty} \) of finite-dimensional spaces \( H_n \subseteq H \) which is complete \( H \). Assume available an orthonormal basis for \( H_n \) consisting of some functions \( \{\varphi_k\}_{k=1}^{n} \) satisfying all boundary conditions of the problem. Following the Galerkin method, we seek an approximate solution to the spectral problem (3) in the form

\[
u_n = \sum_{k=1}^{n} a_k(n) \varphi_k.
\]
Theorem 2.1 Consider a discrete lower semibounded operator $L$ on a separable Hilbert space $H$. If the system of coordinate functions $\{\varphi_k\}_{k=1}^{\infty}$ constitutes a basis for $H$ then the Galerkin method constructed from this system of functions and applied to the problem of finding the eigenvalues of the spectral problem (3) converges.

Theorem 2.2 Consider a discrete lower semibounded operator $L$ acting on a separable Hilbert space $H$. If the system of coordinate functions $\{\varphi_k\}_{k=1}^{\infty}$ constitutes an orthonormal basis for $H$ then

$$\tilde{\mu}_k(n) = (L\varphi_n, \varphi_n) + \delta_n,$$

where $\delta_n = \sum_{k=1}^{n-1} [\tilde{\mu}_k(n-1) - \tilde{\mu}_k(n)]$ and $\tilde{\mu}_k(n)$ is the $n$th Galerkin approximation to the corresponding eigenvalue $\mu_k$ of $L$.

Proof. Inserting (4) into (3) yields

$$\sum_{k=1}^{n} a_k(n) L\varphi_k = \bar{\mu}(n) \sum_{k=1}^{n} a_k(n) \varphi_k.$$

The coefficients $\{a_k(n)\}_{k=1}^{n}$ are determined from the requirement that the left-hand side here be orthogonal to the functions $\{\varphi_l\}_{l=1}^{n}$, which leads to the system of linear equations

$$\sum_{k=1}^{n} a_k(n) \left\{ \bar{\mu}(n) \delta_{k,l} - (L\varphi_k, \varphi_l) \right\} = 0, \quad l = 1, n$$

on the coefficients $\{a_k(n)\}_{k=1}^{n}$, where $\delta_{k,l}$ is the Kronecker symbol. Setting its determinant equal to zero, we arrive at the equation

$$\det \left( A - \bar{\mu}(n) E \right) = 0,$$

which defines the approximate values of the first $n$ eigenvalues $\{\tilde{\mu}_k(n)\}_{k=1}^{n}$ of $L$. Here $E$ is the $n \times n$ identity matrix and $A = (a_{kl})_{k,l=1}^{n}$ is the $n \times n$ matrix with $a_{kl} = (L\varphi_k, \varphi_l)$.

It is known that the eigenvalues $\{\tilde{\mu}_k(n)\}_{k=1}^{n}$ of $A$ satisfy

$$\sum_{k=1}^{n} \tilde{\mu}_k(n) = SpA,$$

which yields

$$\sum_{k=1}^{n} \tilde{\mu}_k(n) = \sum_{k=1}^{n} a_{kk}.$$
Introducing \( \mu_k = \bar{\mu}_k(n) + \varepsilon_k(n) \), we have
\[
\sum_{k=1}^{n} \mu_k = \sum_{k=1}^{n} [a_{kk} + \varepsilon_k(n)].
\] (7)

Subtracting (6) for \( n - 1 \), namely,
\[
\sum_{k=1}^{n-1} \mu_k = \sum_{k=1}^{n-1} [a_{kk} + \varepsilon_k(n - 1)],
\] (8)
from (7), we infer that
\[
\bar{\mu}_n(n) = (L\varphi_n, \varphi_n) + \sum_{k=1}^{n-1} [\bar{\mu}_k(n - 1) - \bar{\mu}_k(n)].
\]

This justifies the theorem.

Observe that to obtain (5) we used the diagonal elements \( a_{kk} = (L\varphi_k, \varphi_k) \) for \( k = 1, n \) of the square matrix \( A = (a_{kl})_{k,l=1}^{n} \). For small \( n \) the error of finding the eigenvalues \( \{\bar{\mu}_k\}_{k=1}^{n} \) can be significant; consequently, we should apply (5) with care. If the requirements of Theorem 1 are fulfilled then the Galerkin method converges; therefore, as \( n \) grows, the calculation of \( \bar{\mu}_n \) using (5) becomes more accurate. In addition, we can calculate the approximate eigenvalues \( \tilde{\mu}_n \) of \( L \) using (5) starting at an arbitrary desired index \( n \) since the values with smaller indices are avoided.

It can be shown that if \( L = T + P \) and \( \|P\| < 0, 5|\lambda_{n+\nu_n} - \lambda_n| \), \( \forall n \in \mathbb{N} \), then formulas (1) and (5) coincide.

3 Numerical experiments

We applied our method to calculate by (5) the eigenvalues of the spectral problem
\[
\begin{cases}
Lu \equiv -u'' + P(x)u = \mu u, & a < x < b; \\
\cos \alpha \ u'(a) + \sin \alpha \ u(a) = 0; \\
\cos \gamma \ u'(b) + \sin \gamma \ u(b) = 0, & \alpha, \gamma \in [0, 2\pi],
\end{cases}
\] (9)

Here \( L \) is an operator acting in \( L_2[a,b] \), and operator \( P \) is bounded. In this case \( L \) is a discrete lower semibounded operator.

Let \( \{\varphi_k\}_{k=1}^{n} \) be an orthonormal system of coordinate functions, satisfying (9).

\( T \) is a selfadjoint operator whose eigenvalues \( \{\lambda_k\}_{k=1}^{\infty} \) are the roots of the transcendental equation
\[
[\sin \alpha \sin(\sqrt{\lambda}a) + \sqrt{\lambda} \cos \alpha \cos(\sqrt{\lambda}a)][\sin \gamma \cos(\sqrt{\lambda}b) - \sqrt{\lambda} \cos \gamma \sin(\sqrt{\lambda}b)] +
\]
Calculation of eigenvalues of discrete lower semibounded operators

\[ +[\sqrt{\lambda} \cos \alpha \sin(\sqrt{\lambda} a) - \sin \alpha \cos(\sqrt{\lambda} a)][\sin \gamma \sin(\sqrt{\lambda} b) + \sqrt{\lambda} \cos \gamma \cos(\sqrt{\lambda} b)] = 0. \]

The corresponding eigenfunctions \( \varphi_k \) are

\[ \varphi_k(s) = C_k \{ [\sin \alpha \sin(\sqrt{\lambda_k} a) + \sqrt{\lambda_k} \cos \alpha \cos(\sqrt{\lambda_k} a)] \cos(\sqrt{\lambda_k} s) + \\
+ [\sqrt{\mu_k} \cos \alpha \sin(\sqrt{\lambda_k} a) - \sin \alpha \cos(\sqrt{\lambda_k} a)] \sin(\sqrt{\lambda_k} s) \}, \quad k = 1, \infty. \]

We can determine the constants \( C_k \) from the normalization condition.

Let us compare the results calculating the eigenvalues \( \tilde{\mu}_k(n) \) of the Sturm–Liouville spectral problem (9) using (5) and the Galerkin method. Denote them by \( \tilde{\mu}_k(n) \) and \( \hat{\mu}_k(n) \) respectively.

**Experiment 1.** Let \( P \) be an operator of multiplication by the function \( p \).

Table 1 presents an example of calculating the eigenvalues of (9) for \( a = 1, b = 3, \alpha = \pi/5, \gamma = \pi/7, p(x) = x^2 - 10x + 11 + (3x^2 - 10x + 9)i \).

We made the calculations on assuming that \( \tilde{\mu}_k(n) - \hat{\mu}_k(n - 1) = 0 \) for \( k = 1, 51, n = 51 \).

**Table 1**

| \( k \) | \( \tilde{\mu}_k(51) \) | \( \hat{\mu}_k(51) \) | \( |\hat{\mu}_k(51) - \hat{\mu}_k(51)| \) |
|--------|-----------------|-----------------|-----------------|
| 1      | \(-3.745674 + 2.940862i\) | \(-4.310179 + 3.541650i\) | \(8.243854 \cdot 10^{-1} \) |
| 2      | \(4.637443 + 2.247669i\) | \(4.802715 + 1.985491i\) | \(3.100431 \cdot 10^{-1} \) |
| 3      | \(17.153279 + 2.110610i\) | \(17.260760 + 2.002740i\) | \(1.522638 \cdot 10^{-1} \) |
| 4      | \(34.487259 + 2.062302i\) | \(34.553462 + 2.006407i\) | \(8.664421 \cdot 10^{-2} \) |
| ...    | ...             | ...             | ...             |
| 12     | \(350.385311 + 2.006931i\) | \(350.392348 + 2.001575i\) | \(8.844251 \cdot 10^{-3} \) |
| 13     | \(412.071642 + 2.005900i\) | \(412.077633 + 2.001356i\) | \(7.522824 \cdot 10^{-3} \) |
| 14     | \(478.692507 + 2.005093i\) | \(478.697670 + 2.001177i\) | \(6.480353 \cdot 10^{-3} \) |
| 15     | \(550.247973 + 2.004436i\) | \(550.252465 + 2.001032i\) | \(5.638965 \cdot 10^{-3} \) |
| ...    | ...             | ...             | ...             |
| 31     | \(2366.259503 + 2.001039i\) | \(2366.260553 + 2.000234i\) | \(1.313989 \cdot 10^{-3} \) |
| 32     | \(2521.705853 + 2.000975i\) | \(2521.706839 + 2.000234i\) | \(1.233034 \cdot 10^{-3} \) |
| 33     | \(2682.086999 + 2.000917i\) | \(2682.087925 + 2.000220i\) | \(1.159348 \cdot 10^{-3} \) |
| 34     | \(2847.402940 + 2.000815i\) | \(2847.403813 + 2.000208i\) | \(1.092060 \cdot 10^{-3} \) |
| ...    | ...             | ...             | ...             |
| 48     | \(5679.979950 + 2.000433i\) | \(5679.980371 + 2.000116i\) | \(5.272798 \cdot 10^{-4} \) |
| 49     | \(5919.317870 + 2.000416i\) | \(5919.318364 + 2.000043i\) | \(6.123044 \cdot 10^{-4} \) |
| 50     | \(6163.590609 + 2.000399i\) | \(6163.590374 + 2.000449i\) | \(2.528352 \cdot 10^{-4} \) |
| 51     | \(6412.798139 + 2.000384i\) | \(6412.818701 + 1.985049i\) | \(2.565019 \cdot 10^{-2} \) |

It is clear from Table 1 that as the index \( k \) of the eigenvalue grows, the corresponding quantities \( |\tilde{\mu}_k(n) - \hat{\mu}_k(n)| \) decrease, with the exception of the last
row for $k = 51$. The jump of value $[\bar{\mu}_{51}(51) - \hat{\mu}_{51}(51)]$ in Table 1 is connected with computing errors when we find $\hat{\mu}_{51}(51)$ by Galerkin method.

**Experiment 2.** Let $P$ be an integral operator:

$$Pu \equiv \int_{a}^{b} g(s)u(s)ds, \ g \in L_2[a, b].$$

Table 2 presents an example of calculating the eigenvalues of (9) for $a = 1$, $b = 3$, $\alpha = \pi / 5$, $\gamma = \pi / 7$, $g(s) = s^2 - 10s + (s^4 + 31s^2 - 10)i$. We made the calculations on assuming that $\bar{\mu}_k(n) - \mu_k(n-1) = 0$ for $k = 1, 71, n = 71$. It is clear from Table 2 that as the index $k$ of the eigenvalue grows, the corresponding quantities $|\bar{\mu}_k(n) - \mu_k(n)|$ decrease.

**Table 2**

| $k$ | $\bar{\mu}_k(71)$ | $\bar{\mu}_k(71)$ | $|\bar{\mu}_k(71) - \hat{\mu}_k(71)|$ |
|-----|------------------|------------------|----------------------------------|
| 1   | 5,379518 + 21,624568i | 5,342937 + 21,580928i | 5,694416 \cdot 10^{-2} |
| 2   | 37,536748 - 0,204934i | 37,605698 - 0,132883i | 1,44705 \cdot 10^{-1} |
| 3   | 86,813778 + 0,227059i | 86,762476 + 0,199216i | 5,837055 \cdot 10^{-2} |
| 4   | 155,916154 + 0,012057i | 155,917850 - 0,011658i | 1,742831 \cdot 10^{-3} |
| \ldots | \ldots | \ldots | \ldots |
| 12  | 1419,213558 - 0,000146i | 1419,213560 - 0,000146i | 2,260431 \cdot 10^{-6} |
| 13  | 1665,953372 + 0,000628i | 1665,953364 + 0,000625i | 8,31796 \cdot 10^{-6} |
| 14  | 1932,432579 - 0,000079i | 1932,432590 - 0,000079i | 8,944562 \cdot 10^{-7} |
| 15  | 2218,650922 + 0,000354i | 2218,650919 + 0,000353i | 3,518841 \cdot 10^{-6} |
| \ldots | \ldots | \ldots | \ldots |
| 31  | 9482,679042 + 0,000019i | 9482,679042 + 0,000019i | 4,499317 \cdot 10^{-8} |
| 32  | 10104,464107 - 0,00003i | 10104,464107 - 0,00003i | 6,241168 \cdot 10^{-9} |
| 33  | 10745,988379 + 0,000015i | 10745,988379 + 0,000015i | 3,091531 \cdot 10^{-8} |
| 34  | 11407,251864 - 0,000024i | 11407,251864 - 0,000024i | 4,337437 \cdot 10^{-9} |
| \ldots | \ldots | \ldots | \ldots |
| 48  | 22737,557620 - 0,000001i | 22737,557620 - 0,000001i | 5,475590 \cdot 10^{-10} |
| 49  | 23694,909243 + 0,000003i | 23694,909243 + 0,000003i | 2,882942 \cdot 10^{-9} |
| 50  | 24672,000075 - 0,000000i | 24672,000075 - 0,000000i | 4,285883 \cdot 10^{-10} |
| 51  | 25668,830116 + 0,000003i | 25668,830116 + 0,000003i | 2,267648 \cdot 10^{-9} |

Numerous calculations for various values of the parameters $a, b, c, d, \alpha, \beta, p, g$ demonstrated high accuracy and computational efficiency of our formula (5) for the eigenvalues of the spectral problem (9).

**References**

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