A Mathematical Model for Transmission of Dengue

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Abstract

In this paper we propose a mathematical model based on a non-linear system of ordinary differential equations that represent the dynamic of transmission of dengue. It is done a complete stability analysis of the model proposed about the solution of equilibrium, on the base of population growth number of the Aedes aegypti mosquito and the basic reproductive number.

Keywords: Mathematical Model, Basic Reproductive Number, Population
1 Introduction

Dengue is a viral disease produced by the virus (DENV) that is transmitted to humans through the bite of the Aedes aegypti mosquito [3]. The transmission cycle starts when an infected Aedes mosquito bites a healthy person, as a result, this human host may infect a susceptible mosquito after it has taken a blood meal from that person.

The life cycle of the Aedes aegypti has two phases: the aquatic one (immature phase) and the aerial one (mature phase). In the first one, there are three stages: the egg, the larva (with four evolving phases) and the pupae; while in the second phase the mosquito becomes an adult. The dengue virus is transmitted to humans only by female mosquitoes, since they bite with the target of maturating their eggs and collecting sources of alternate energies. On the other hand, the male mosquitoes feed themselves from plant’s nectars [3].

2 The model

From the model proposed, we assumed that human population is constant, that is to say, the growth rate is equal to death rate. The variables and parameters that are used for the model proposed are described in the Table 1 and their transmission dynamics in the Figure 1.

The proposed model combines three uncoupled systems as follows. The system (1) represents the logistic growth of the breeding sites.

\[
\dot{c}_i = \gamma_i c_i \left( 1 - \frac{c_i}{k_i} \right), \quad i = 1, 2, 3.
\]

(1)

The system (2) represents the mosquito population growth Aedes aegypti considering its evolving phases.

\[
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{z}
\end{bmatrix} =
\begin{bmatrix}
\delta z - \epsilon x \\
\sigma z - \epsilon y \\
\phi \rho y \left( 1 - \frac{z}{\sum a_i c_i + b} \right) - (\delta + \sigma + \nu) z
\end{bmatrix}
\]

(2)
The system (3) represent the virus transmission in human population.

\[
\begin{align*}
\dot{S} &= \mu N - \beta \psi \frac{L}{N} \frac{y}{x+y} S - \mu S \\
\dot{I} &= \beta \psi \frac{L}{N} \frac{y}{x+y} S - (\theta + \mu) I \\
\dot{R} &= \theta I - \mu R
\end{align*}
\]
3 Results

The solution to the system (1) with initial conditional \( c_i(0) = c_{i0} \), is of the form

\[
    c_i(t) = \frac{\kappa_i c_{i0}}{(\kappa_i - c_{i0})e^{-\gamma_i t} + c_{i0}}, \quad i = 1, 2, 3.
\]

Since \( \gamma_i > 0 \),

\[
    \lim_{t \to +\infty} c_i(t) = \kappa_i
\]

Therefore, the system (1) has an equilibrium point asymptotically stable with coordinates \( (\kappa_1, \kappa_2, \kappa_3) \). Thus,

\[
    \lim_{t \to +\infty} \left( \sum a_i c_i(t) + b \right) = \sum a_i \kappa_i + b
\]

doing \( K = \sum a_i \kappa_i + b \) and replaced in the third equation of the system (2), we obtain the system:

\[
    \begin{align*}
        \dot{x} &= \delta z - \epsilon x \\
        \dot{y} &= \sigma z - \epsilon y \\
        \dot{z} &= \phi \rho y \left( 1 - \frac{z}{K} \right) - (\delta + \sigma + \nu) z
    \end{align*}
\]
Algebraic calculations show that the equilibrium points of the system (4) are:

\[ Q_1 = (0, 0, 0) \quad \text{and} \quad Q_2 = \left( \frac{\delta K(h - 1)}{\epsilon h}, \frac{\sigma K(h - 1)}{\epsilon h}, \frac{K(h - 1)}{h} \right), \]

where \( h = \frac{\phi \rho \sigma}{\epsilon (\delta + \sigma + \nu)} \) represents the population growth number of the Aedes aegypti mosquito.

**Theorem 3.1** (\( Q_2 \) local stability)

If \( h > 1 \), \( Q_2 \) is locally asymptotically stable.

**Proof 3.1** The Jacobian matrix associated to the linearisation of the system (4) is

\[
J(x, y, z) = \begin{pmatrix}
-\epsilon & 0 & \delta \\
0 & -\epsilon & \sigma \\
0 & \phi \rho \left(1 - \frac{z}{R}\right) & -\frac{\phi \rho}{R} y - (\delta + \sigma + \nu)
\end{pmatrix}
\]

When evaluating the matrix \( J \) in the equilibrium point \( Q_2 \), we obtain:

\[
J(Q_2) = \begin{pmatrix}
-\epsilon & 0 & \delta \\
0 & -\epsilon & \sigma \\
0 & \frac{\phi \rho}{h} & -(\delta + \sigma + \nu)h
\end{pmatrix}
\]

her characteristic polynomial can be written as:

\[
p(\lambda) = (\lambda + \epsilon) \left( \lambda^2 + (\epsilon + (\delta + \sigma + \nu)h) \lambda + \epsilon (\delta + \sigma + \nu)(h - 1) \right)
\]

In this case, if \( h > 1 \), the real part of all eigenvalues of the matrix \( J(Q_2) \) is negative, since they reach the established condition in Routh-Hurwitz in [4]. Therefore, the equilibrium point \( Q_2 \) is locally asymptotically stable.

This means that if \( h > 1 \), the population of the mosquito grows exponentially to equilibrium:

\[
x = \frac{\delta K(h - 1)}{\epsilon h}, \quad y = \frac{\sigma K(h - 1)}{\epsilon h} \quad \text{and} \quad z = \frac{K(h - 1)}{h}
\]

respectively. Substituting these results in the system (1) we obtain,

\[
\begin{aligned}
\dot{S} &= \mu N - \beta \psi \frac{L}{N} \frac{\sigma}{\delta + \sigma} S - \mu S \\
\dot{I} &= \beta \psi \frac{L}{N} \frac{\sigma}{\delta + \sigma} S - (\theta + \mu) I \\
\dot{R} &= \theta I - \mu R
\end{aligned}
\]

doing,

\[
p = \frac{S}{N}, \quad q = \frac{I}{N} \quad \text{and} \quad r = \frac{R}{N}.
\]
we obtain the system
\[
\begin{align*}
\dot{p} &= \mu(1 - p) - \beta \psi \frac{\sigma}{\delta + \sigma} pq \\
\dot{q} &= \beta \psi \frac{\sigma}{\delta + \sigma} pq - (\theta + \mu)q
\end{align*}
\]  
(5)
defined in the set of biological interest
\[
\Omega = \{(p, q) \in \mathbb{R}^2 : p \geq 0, q \geq 0, p + q \leq 1\}
\]
The equilibrium points of the system (5) are
\[
E_1 = (1, 0) \quad \text{and} \quad E_2 = \left(1, \frac{\mu \left( R_0 - 1 \right)}{R_0, R_0(\theta + \mu)} \right)
\]
where \( R_0 = \frac{\beta \psi \sigma}{(\delta + \sigma)(\theta + \mu)} \) represents the basic reproductive number of the disease. This represents the expected number of secondary cases produced in a completely susceptible population, by a infected mosquito.

**Theorem 3.2** (\( E_1 \) and \( E_2 \) local stability)

- If \( R_0 < 1 \), \( E_1 \) is locally asymptotically stable in \( \Omega \).
- If \( R_0 > 1 \), \( E_2 \) is locally asymptotically stable in \( \Omega \).

**Proof 3.2** The Jacobian matrix associated to the linearisation of the system (5) is
\[
J(p, q) = \begin{pmatrix}
-\mu - \beta \psi \frac{\sigma}{\delta + \sigma} q & -\beta \psi \frac{\sigma}{\delta + \sigma} p \\
\beta \psi \frac{\sigma}{\delta + \sigma} q & \beta \psi \frac{\sigma}{\delta + \sigma} p - (\theta + \mu)
\end{pmatrix}
\]
When evaluating the matrix \( J \) in the equilibrium point \( E_1 \), we obtain:
\[
J(E_1) = \begin{pmatrix}
-\mu & -\beta \psi \frac{\sigma}{\delta + \sigma} \\
0 & (\theta + \mu)(R_0 - 1)
\end{pmatrix}
\]
her characteristic polynomial can be written as:
\[
p(\lambda) = (\lambda + \mu)(\lambda + (\theta + \mu)(1 - R_0))
\]
If \( R_0 < 1 \), the real part of all eigenvalues of the matrix \( J(E_1) \) is negative, therefore, the equilibrium point \( E_1 \) is locally asymptotically stable.

Now, evaluating the matrix \( J \) in the equilibrium point \( E_2 \), we obtain:
\[
J(E_2) = \begin{pmatrix}
-\mu R_0 & -(\theta + \mu) \\
\mu(R_0 - 1) & 0
\end{pmatrix}
\]
her characteristic polynomial can be written as:
\[
p(\lambda) = \lambda^2 + \mu R_0 \lambda + \mu(\theta + \mu)(R_0 - 1)
\]
In this case, if $R_0 > 1$, the real part of all eigenvalues of the matrix $J(E_2)$ is negative, since they reach the established condition in Routh-Hurwitz in [4]. Therefore, the equilibrium point $E_2$ is locally asymptotically stable.

**Proposition 3.1** The system (5) does not have periodic (solutions) orbits in $\Omega$.

**Proof 3.1** Being

$$f(p,q) = \mu(1-p) - \beta\psi \frac{\sigma}{\delta + \sigma}pq \quad \text{and} \quad g(p,q) = \beta\psi \frac{\sigma}{\delta + \sigma}pq - (\theta + \mu)q$$

two multi variable functions that represent the right side of the system (5) and $l(p,q) = \frac{1}{(\theta + \mu)q}$ a differentiable continuous function for everything $q \neq 0$ in $\Omega$. Then, applying Dulac’s criterion ([1], Theorem 4.8, p. 153) we obtain:

$$\frac{\partial}{\partial p} (l(p,q)f(p,q)) + \frac{\partial}{\partial q} (l(p,q)g(p,q)) = \frac{\partial}{\partial p} \left( \frac{\mu(1-p)}{(\theta + \mu)q} - R_0p \right) + \frac{\partial}{\partial q} (R_0p - 1)$$

$$= -\frac{\mu}{(\theta + \mu)q} - R_0$$

this is negative for all $(p,q) \in \Omega$ with $q \neq 0$. Thus, the system (5) does not have periodic solutions.

Now, if $q = 0$, the system (5) is written as:

$$\begin{cases} \dot{p} & = \mu(1-p) \\ \dot{q} & = 0 \end{cases}$$

with $(p_0, 0) \in \Omega$. Thus,

$$p(t) = 1 - (1 - p_0)e^{-\mu t}$$

$$q(t) = 0$$

which is not a periodic solution.

Because of the system (5) does not have periodic solution in $\Omega$, we make a study of the global stability of the equilibrium points.

**Theorem 3.3** ($E_1$ global stability)

If $R_0 \leq 1$, the equilibrium point $E_1$ is globally stable.

**Proof 3.3** Being $V(p,q) = \frac{r}{\theta + \mu}$ a real value function of class $C^1(\Omega)$ such that:

- $V(E_1) = 0$
- $V(p,q) \geq 0$ for all $(p,q) \in \Omega - E_1$
• Deriving $V$ with respect to $t$, we obtain

$$
\dot{V} = \frac{\dot{q}}{\theta + \mu} = (R_0 p - 1)q
$$

as $R_0 \leq 1$ and $0 \leq p \leq 1$, so $R_0 p \leq 1$ and therefore, $\dot{V} \leq 0$.

In this way, according [6] (Theorem 1.1.2, p. 12) $V(p, q)$ represents a Lyapunov's function for the equilibrium $E_1$ with $R_0 \leq 1$. Thus, $E_1$ is globally stable under this condition.

Now, being

$$S = \left\{ (p, q) \in \Omega : \dot{V}(p, q) = 0 \right\} = \left\{ (p, q) \in \Omega : q = 0 \right\}$$

the set where the orbital derived from $V$ is zero. So:

• If $R_0 < 1$, $\dot{V} = 0$ when $q = 0$.

• If $R_0 = 1$, $\dot{V} = 0$ when $p = 1$ or $q = 0$.

Then, in both cases the most invariant set of $S$ is

$$M = \{E_1\}$$

In this way, for the LaSalle’s theorem ([2], Theorem 4.4, p. 128), the equilibrium $E_1$ is globally asymptotically stable when $R_0 \leq 1$.

Figure 2 shows the vectorial field of (5) for $R_0 \approx 0.89$, where it is observed that all the solutions with initial condition in $\Omega$ approximated to the equilibrium point $E_1$.

![Figure 2: Vectorial field of (5) with $R_0 \leq 1$](image-url)
**Theorem 3.4 (E₂ global stability)**

If \( R_0 > 1 \), the equilibrium point \( E_2 \) is globally stable.

**Proof 3.4** Being \( E_2 = (p^*, q^*) \) the equilibrium point of the system (5), with

\[
p^* = \frac{1}{R_0} \quad \text{and} \quad q^* = \frac{\mu(R_0 - 1)}{R_0(\theta + \mu)}
\]

and,

\[
W(p, q) = (p - p^*) + (q - q^*) - p^* \ln \frac{p}{p^*} - q^* \ln \frac{q}{q^*}
\]

a real value function of class \( C^1(\Omega) \). So:

\[
\dot{W} = \frac{p}{p} (p - p^*) + \frac{q}{q} (q - q^*)
\]

as

\[
\frac{\dot{p}}{p} = \mu \left( \frac{1 - p}{p} \right) - \beta \psi \frac{\sigma}{\delta + \sigma} q \quad \text{and} \quad \frac{\dot{q}}{q} = \beta \psi \frac{\sigma}{\delta + \sigma} p - (\theta + \mu)
\]

so

\[
\dot{W} = \left( \frac{\mu}{p} - \mu - \beta \psi \frac{\sigma}{\delta + \sigma} q^* \right) (p - p^*) + \left( \beta \psi \frac{\sigma}{\delta + \sigma} p - (\theta + \mu) \right) (q - q^*)
\]

but

\[
-\mu = \beta \psi \frac{\sigma}{\delta + \sigma} p^* \quad \text{and} \quad - (\theta + \mu) = - \beta \psi \frac{\sigma}{\delta + \sigma} p^*
\]

therefore,

\[
\dot{W} = -\mu \frac{(p - p^*)^2}{p^* p} \leq 0
\]

Thus, according [5] \( W(p, q) \) represents a Lyapunov’s function for the equilibrium point \( E_2 \). In this way, if \( R_0 > 1 \), \( E_2 \) is globally stable in \( \Omega \).

Now, being

\[
T = \{ (p, q) \in \Omega : \dot{W}(p, q) = 0 \} = \{ (p, q) \in \Omega : p = p^* \}
\]

the set where the \( W \) orbital derived is zero. So, at the moment of replacing this condition in the system (5), we obtain,

\[
\begin{cases}
\dot{p} = \mu \left( \frac{R_0 - 1}{R_0} \right) - (\theta + \mu) q \\
\dot{q} = 0
\end{cases}
\]
Thus, the most invariant set of $T$ is

$$M = \{E_2\}$$

In this way, for the LaSalle’s theorem ([2], Theorem 4.4, p. 128), the equilibrium $E_2$ is globally asymptotically stable when $R_0 > 1$.

Figure 3 shows the vectorial field of (5) for $R_0 \approx 2.75$, where it is observed that all the solutions with initial condition in $\Omega$ approximated to the equilibrium point $E_2 \approx (0.36, 0.02)$ when the time passes. This means that the disease remains in the environment.

![Figure 3: Campo vectorial de (5) con $R_0 > 1$](image)

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**References**


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