A Note of the Strong Convergence of
the Mann Iteration for Demicontractive Mappings

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Abstract

In [2] an additional condition of quasi-expansive type is given, which ensures the strong convergence of the Mann iteration for a demicontractive mapping in real Hilbert spaces. In this note we further analyse this condition and, for a particular case, it is showed the shape of a mapping which is both demicontractive and quasi-expansive.

Mathematics Subject Classification: 47H10, 65J15

Keywords: Fixed point, Mann iteration, Demicontractive mapping, Strong convergence

1 Introduction

In [2] the author consider the following additional condition (under the name of Condition A), which ensure the strong convergence of a sequence generated by the Mann iteration in the case of a demicontractive mapping $T : C \to C$ ($C$ is a closed convex subset of a real space $\mathcal{H}$):

\textit{Condition A:} The mapping $T : C \to C$ is said to satisfy condition $\mathcal{A}$, if $T$ has a fixed point $p \in C$, $T$ is Fréchet differentiable at $p \in C$ and $I - T'(p)$ is invertible.

Actually $\mathcal{A}$ is not an additional condition, it is a stronger smoothness condition which is required in place of demi-closedness at zero. More precisely, the first result of [2] states that if $T$ is $L$-demicontractive ($0 < L < 1$) and
satisfies condition $\mathcal{A}$, then the Mann iteration with control sequence satisfying $0 < a \leq t_n \leq b < 1 - L$, converges strongly to a fixed point of $T$. The second result states that if $T$ is $(\alpha, L)$-strictly demicontractive and satisfies condition $\mathcal{A}$, then $T_t := (1 - t)I + tT$ is a Kannan contraction provided that $t$ satisfies certain conditions; as a corollary, it is obtained that the Mann iteration with constant control sequence, $t_n = t$, $n = 0, 1, \ldots$ converges strongly to the unique fixed point of $T$.

The proofs of both results are based on the two lemmas.

The first lemma (Lemma 2.1) is a simple variant of Mean Value Theorem and asserts that, if $T$ is differentiable at $p$ then

$$T(u) - T(p) = (T'(p) + R_u)(u - p),$$

where $R_u$ is a linear mapping satisfying the following condition: Given $c > 0$ there exists $r > 0$ such that if $u \in S(p, r) = \{x|\|x - p\| \leq r\}$ then $\|R_u\| \leq c$.

The second lemma (Lemma 2.2) asserts that, if $T$ satisfies the condition (A), then for any $c > 0$ satisfying $cn < 1$, where $\eta = \|I - T'(p)\|$, there exists $r > 0$ such that

$$\|x - p\| \leq \beta \|x - T(x)\|, \forall x \in S(p, r),$$

where $\beta = \frac{\eta}{1 - \eta c}$. If $\beta < 1$ then $T$ satisfies the condition

$$\|x - p\| \leq \frac{\beta}{1 - \beta} \|T(x) - p\|, \forall x \in S(p, r).$$

In [3] is proposed the term quasi-expansive for a mapping satisfying this last inequality. The term is motivated by the fact that around $p$ the mapping $T$ has an expansive behavior; of course, if $\beta/(1 - \beta) > 1$, the Picard iteration does not converge. The quasi-expansivity property is used then as additional condition to prove strong convergence of the Mann iteration (Theorem 3.1 in [2]).

In this note we further analyse the quasi-expansive condition. We complete the proof of Lemma 2.1 from [2] with necessary arguments in Hilbert spaces and show that the new proposed mean value formula is equivalent with the existence of derivative. In the particular case of a real function we show how the shape of a function that is both demicontractive and quasi-expansive should be. We present also several examples showing that the strong demicontractivity and quasi-expansivity are not contradictory.
2 Some remarks on the quasi-expansive mappings

Let $\mathcal{H}$ be a real Hilbert space, scalar product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, $\mathcal{C}$ a closed convex subset of $\mathcal{H}$, $T : \mathcal{C} \rightarrow \mathcal{C}$ a mapping, and $p$ a point in $\mathcal{C}$.

Condition $A$. We say that $T$ satisfies condition $A$ if there exists a linear mapping $D$ (which depends on $p$) and for every $u \in \mathcal{H}$ there exists a linear mapping $R_u$ such that

(i) $T(u) - T(p) = (D + R_u)(u - p)$, $\forall u \in \mathcal{H}$;

(ii) Given $c > 0$ there exists $r > 0$ such that $u \in S(p, r) = \{x | \|x - p\| \leq r \} \Rightarrow \|R_u\| \leq c$.

Lemma 2.1 The condition $A$ is equivalent with the existence of the derivative of $T$ in $p$.

Proof If $A$ is satisfied, then obviously, $T$ is Fréchet differentiable in $p$ and $D = T'(p)$.

Suppose now that $T$ is Fréchet differentiable in $p$ and for $u \in \mathcal{C}$ let $E$ be defined by

$$E(u) = \frac{T(u) - T(p) - T'(p)(u - p)}{\|u - p\|}.$$ 

For any $c > 0$ there exists $r > 0$ such that $u \in S(p, r) \Rightarrow \|E(u)\| \leq c$. Let us define $D = T'(p)$ and

$$R_u(x) = \frac{\langle u - p, x \rangle}{\|u - p\|}E(u).$$

Obvious $R_u$ is a bounded linear mapping and

$$\|R_u\| = \sup_{\|x\| = 1} \frac{\langle u - p, x \rangle}{\|u - p\|} \|E(u)\| \leq \sup_{\|x\| = 1} \frac{\|u - p\|\|x\|}{\|u - p\|} \|E_u\| \leq c.$$ 

We have also

$$T(u) - T(p) = T'(p)(u - p) + T(u) - T(p) - T'(p)(u - p)$$

$$= T'(p)(u - p) + \|u - p\|E(u)$$

$$= T'(p)(u - p) + \frac{\langle u - p, u - p \rangle}{\|u - p\|}E(u) = T'(p)(u - p) + R_u(u - p)$$

$$= (T'(p) + R_u)(u - p).$$

Therefore the conditions (i), (ii) are satisfied. $\square$

Remark The statement of Lemma 2.1 from [2] coincides with the second part of our lemma. No proof is given of Lemma 2.1 from [2], only it is mentioned that the proof is immediate ($R(u)$ is defined in a similar way as the $E(u)$ above). It should be underlined that the proposed formula in [2] is correct only in finite dimensional space.
In the sequel we outline some properties of quasi-expansive mappings.

The first one is that if $T$ is quasi-expansive with $0 < \beta < 1$, then the rate of convergence of the Picard iteration is at most linear. This results immediately from

$$\frac{\|T(x_n) - p\|}{\|x_n - p\|} = \frac{\|x_{n+1} - p\|}{\|x_n - p\|} \geq \frac{1 - \beta}{\beta}. $$

In the case of the Mann iteration, we cannot conclude this with certainty. Indeed, let $T_{t_n}$ be the generation function of Mann iteration, $T_{t_n} = (1 - t_n)I + t_nT$, where the control sequence $\{t_n\}$ satisfies $t_n > \beta$, $n = 0, 1, \ldots$. We have

$$\frac{\|T_{t_n}(x_n) - p\|}{\|x_n - p\|} = \frac{\|x_{n+1} - p\|}{\|x_n - p\|} \geq \frac{t_n - \beta}{\beta}. $$

If $t_n \to \beta$ then the rate of convergence can be superlinear.

**Remark** In the case of Mann iteration, if $T$ is $L$-demicontractive, $0 < L < 1$, and has a unique fixed point $p$, we have $\|x_{n+1} - p\| \leq \|x_n - p\|$. The requirement of convexity of $C$ is necessary to keep the sequence inside of $C$. We can also suppose that $T : C \to \mathcal{H}$, where $C$ is an open subset of $\mathcal{H}$, and impose the condition that some sphere $S(p, r) = \{x : \|x - p\| \leq r\}$ be inside of $C$ and $x_0 \in S(p, r)$. Then from $\|x_{n+1} - p\| \leq \|x_n - p\|$ it follows that $\{x_n\} \subset C$.

An important problem is to estimate the set of mappings satisfying the both conditions, the demicontractivity and quasi-expansivity.

In the particular case of the set $\mathcal{F}$ of real continuous functions, $f : [a, b] \to \mathbb{R}$, having a unique fixed point $p \in [a, b]$, it can be done the following answer. Note first that the condition of demicontractivity is equivalent with the following condition

$$\langle x - T(x), x - p \rangle \geq \lambda \|T(x) - x\|^2, \quad x \in C, \quad p \in \text{Fix}(T),$$

where $\lambda = (1 - L)/2$. Note also that if $T$ is quasi-expansive then $T$ has a unique fixed point $p$ (this is the reason that we consider those functions in $\mathcal{F}$ which have each a unique fixed point).

Let $\mathcal{DC}_\lambda \subset \mathcal{F}$ denote the set of demicontractive functions and $\mathcal{QE}_\beta \subset \mathcal{F}$ the set of quasi-expansive functions.

**Theorem 2.2** If $\beta > \lambda$ then $\mathcal{DC}_\lambda \cap \mathcal{QE}_\beta \neq \emptyset$ and for any $f \in \mathcal{DC}_\lambda$ there exists $\beta$ such that $f \in \mathcal{QE}_\beta$; thus any demicontractive real function is also quasi-expansive.

**Proof** It is elementary to see that $f$ is demicontractive if and only if

$$\begin{cases} x < f(x) < x - 1/\lambda(x - p), & \text{if } x < p, \\ x - 1/\lambda(x - p) < f(x) < x, & \text{if } x > p. \end{cases}$$
The graph of \( f \in \mathcal{DC}_\lambda \) must lie in the shaded region in Figure 1a, i.e., in the region \( \mathcal{D}_1 \cup \mathcal{D}_2 \).

In a similar way, \( f \in \mathcal{QE}_\beta \) is quasi-expansive if and only if

\[
\begin{cases}
  f(x) > x - 1/\beta(x - p), & \text{if } x < p, \\
  f(x) < x - 1/\beta(x - p), & \text{if } x > p.
\end{cases}
\]

The graph of \( f \in \mathcal{QE}_\beta \) must lie in the shaded region in Figure 1b, i.e., in the region \( \mathcal{E}_1 \cup \mathcal{E}_2 \).

Therefore \( f \) is both demicontractive and quasi-expansive if the graph of \( f \) belongs to the intersection of the two regions, \( \mathcal{D}_1 \cup \mathcal{D}_2 \) and \( \mathcal{E}_1 \cup \mathcal{E}_2 \), i.e., in the shaded region in Figure 1c.

![Figure 1: (a) The demicontractive regions; (b) The quasi-expansive regions; (c) The demicontractive and quasi-expansive regions.](image)

It is obvious that \( \mathcal{D}_1 \cap \mathcal{E}_1 \neq \emptyset \) and \( \mathcal{D}_2 \cap \mathcal{E}_2 \neq \emptyset \) if and only if \( \beta > \lambda \). Geometrically, this means that the slope of \( y = x - (1/\beta)(x - p) \) is greater than the slope of \( y = x - (1/\lambda)(x - p) \). Now let \( f \) be in \( \mathcal{DC}_\lambda \). Because \( f \) has \( p \) as the unique fixed point in \([a, b] \), there exists a \( \beta_m \) sufficiently large such that the straight line \( y = x - (1/\beta_m)(x - p) \) to be below the graph of \( f \) if \( x < p \) and above it if \( p > p \). It follows that \( f \in \mathcal{QE}_{\beta_m} \).

**Remark** As a consequence of Theorem 2.2 it follows that in our particular case the conditions of demicontractivity and continuity are sufficient for the convergence of the Mann iteration (with the already stated assumptions on the control sequence). This is in accordance with Theorem 1 [4] which states that in real Hilbert spaces the Mann iteration converges weakly to a fixed point of \( T \) provided that \( T \) is demicontractive and demiclosed at zero.
3 Numerical examples

This section is devoted to present some numerical examples of functions that satisfy both demicontractivity and quasi-expansivity condition and, in the same time, they are not quasi-nonexpansive.

**Example 3.1** Let \( f : \mathbb{R} \to \mathbb{R} \) defined by

\[
 f(x) = \begin{cases} 
 2 & \text{if } x < 2/7, \\
 -1.4(x - 1) + 1 & \text{if } 2/7 \leq x \leq 12/7, \\
 0 & \text{if } x > 12/7. 
\end{cases}
\]

This function has a unique fixed point \( p = 1 \), is demicontractive \( (\alpha = 0.81, L = 0.2) \) and quasi-expansive \( (\beta = 0.5) \), and is not quasi-nonexpansive. Note that Mann iteration converges to \( p = 1 \) if \( t < 0.8333... \) The Theorem 3.1 from [2] gives \( 1 - L = 0.8 \) for the superior bound of \( t \) which is a satisfactory estimation.

**Example 3.2** Let \( f : [0.2, 0.4] \to [0.2, 0.4] \) defined by

\[
 f(x) = \begin{cases} 
 0.4 & \text{if } x < s_1, \\
 -0.4x^2 - 1.6x + 0.8 & \text{if } x \in [s_1, s_2], \\
 0.2 & \text{if } x > s_2 
\end{cases}
\]

where \( s_1 = 0.236066... \), \( s_2 = 0.345649... \). This function has \( p = 0.29436... \) as fixed point, is demicontractive \( (L = 0.33) \), and quasi-expansive \( (\beta = 0.52) \). The function is not quasi-nonexpansive.

**Example 3.3** Let \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) defined by \( f(x) = Ax + b \) where

\[
 A = \begin{pmatrix} 
 0.8 & -0.7 \\
 0.7 & 0.85 
\end{pmatrix}, \quad b = \begin{pmatrix} 
 -1 \\
 2 
\end{pmatrix}
\]

The mapping \( f \) has the fixed point \( p = (-2.981..., -0.577...) \), is both demicontractive \( (L = 0.88) \) and quasi-expansive \( (\beta = 1.5) \); it is not quasi-nonexpansive.

4 Conclusion

The quasi-expansive condition seems to be one of the most reasonable additional condition that ensure the strong convergence of the Mann iteration for demicontractive mappings. In the case of real function, besides of the common convergence assumptions (demicontractivity and demiclosednes at zero), it does not involve any other condition.
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References


Received: November 29, 2015; Published: January 19, 2016