Analysis of a Stochastic SIRS Epidemic Model with Specific Functional Response

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Abstract

A new stochastic SIRS epidemic model with specific functional response is proposed and analyzed. First, we show that the model is biologically well-posed by proving the global existence, positivity and boundedness of solutions. Moreover, sufficient conditions for the extinction and persistence of the disease are also obtained. In the end, some numerical simulations are presented to illustrate our analytical results.

Keywords: Stochastic SIRS model, specific functional response, extinction, persistence

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1 Introduction

In this paper, we consider the following SIRS epidemic model with specific functional response:

\[
\begin{align*}
\frac{dS}{dt} &= \left( A - \mu_1 S - \frac{\beta SI}{1 + \alpha_1 S + \alpha_2 I + \alpha_3 SI} + \gamma R \right) dt, \\
\frac{dI}{dt} &= \left( \frac{\beta SI}{1 + \alpha_1 S + \alpha_2 I + \alpha_3 SI} - (\mu_2 + r) I \right) dt, \\
\frac{dR}{dt} &= \left( r I - (\mu_3 + \gamma) R \right) dt,
\end{align*}
\]

where \( S(t), I(t) \) and \( R(t) \) denote the numbers of susceptible, infective, and recovered individuals at time \( t \), respectively. \( A \) is the recruitment rate of the population, \( \mu_1, \mu_2 \) and \( \mu_3 \) are the deaths rates of \( S, I \) and \( R \), respectively. It is natural biologically to assume that \( \mu_1 \leq \min\{\mu_2, \mu_3\} \). The parameter \( \gamma \) is the rate at which recovered individuals lose immunity and return to the susceptible class and \( r \) is the recovery rate of the infective individuals. The incidence rate of disease is modeled by the specific functional response \( \frac{\beta SI}{1 + \alpha_1 S + \alpha_2 I + \alpha_3 SI} \), where \( \beta \) is the infection coefficient and \( \alpha_1, \alpha_2, \alpha_3 \geq 0 \) are constants. This specific functional response was introduced by Hattaf et al. (see Section 5 in [3]) and used in [1, 10, 2, 13], which covers various types of incidence rate existing in the literature such as the mass action called also the bilinear incidence rate; the saturation incidence; Beddington-DeAngelis response and Crowley-Martin response.

On the other hand, we assume that the parameters \( \beta, \mu_1, \mu_2 \) and \( \mu_3 \) are subject to random fluctuations. Therefore, the system (1) becomes

\[
\begin{align*}
\frac{dS}{dt} &= \left( A - \mu_1 S - \frac{\beta SI}{\psi(S, I)} + \gamma R \right) dt - \sigma_1 S dB_1 - \sigma_4 \frac{\beta SI}{\psi(S, I)} dB_4, \\
\frac{dI}{dt} &= \left( \frac{\beta SI}{\psi(S, I)} - (\mu_2 + r) I \right) dt - \sigma_2 I dB_2 + \sigma_4 \frac{\beta SI}{\psi(S, I)} dB_4, \\
\frac{dR}{dt} &= \left( r I - (\mu_3 + \gamma) R \right) dt - \sigma_3 R dB_3,
\end{align*}
\]

where \( \psi(S, I) = 1 + \alpha_1 S + \alpha_2 I + \alpha_3 SI \) and \( \sigma_i \) denotes the intensity of the white noise, \( i = 1, 2, 3, 4 \). \( B_1, B_2, B_3 \) and \( B_4 \) are independent Brownian motions defined on a complete probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) space with a filtration \( \{\mathcal{F}_t\}_{t \geq 0} \) satisfying the usual conditions, i.e., it is right continuous and increasing while \( \mathcal{F}_0 \) contains all \( \mathbb{P} \)-null sets.

The importance of our stochastic model (2) is that it includes many special cases existing in the literature. For example, when \( \alpha_1 = \alpha_2 = \alpha_3 = 0, \gamma = 0, A = \mu_1 = \mu_2 = \mu_3 \) and \( \sigma_1 = \sigma_2 = \sigma_3 = 0 \), we obtain the stochastic epidemic model with bilinear incidence rate introduced by Tornatore et al. in [15]. When
\(\alpha_1 = \alpha_2 = \alpha_3 = 0, A = \mu_1 = \mu_3, \sigma_1 = \sigma_2 = \sigma_3 = 0,\) and \(\mu_2 = \mu_1 + d,\) where \(d\) is death rate due to disease, we obtain the model proposed by Lu [11]. When \(\alpha_1 = \alpha_2 = \alpha_3 = 0, \mu_1 = \mu_3, \mu_2 = \mu_1 + d, \gamma = 0,\) and \(\sigma_1 = \sigma_2 = \sigma_3 = 0,\) we get the model of Ji et al. [4]. When \(\alpha_1 = \alpha_3 = 0, \mu_1 = \mu_3, \mu_2 = \mu_1 + d\) and \(\gamma = 0,\) and \(\sigma_1 = \sigma_2 = \sigma_3 = 0,\) we obtain the model of Lin et al. [8]. Also, the stochastic SIR epidemic model with Beddington-DeAngelis functional response proposed by Tan and Guo [14] is a particular case of (2) when \(\alpha_3 = 0\) and \(\gamma = 0.\) In addition, we obtain the stochastic SIRS epidemic model with Beddington-DeAngelis functional response given in [7] when \(\alpha_3 = 0\) and \(\sigma_4 = 0.\)

The rest of the paper is organized as follows. In the next section, firstly, we prove that our generalized stochastic model (2) is biologically well-posed by showing the global existence, positivity and boundedness of solutions. Sufficient conditions for the extinction and persistence of the disease are established in Sections 3 and 4, respectively. In Section 5, we present some numerical simulations to illustrate our main results. The paper ends with a brief discussion and conclusion in Section 6.

2 Global existence and positivity of the solution

In this section, we establish the global existence, positivity, and boundedness of solutions of system (2) because this model describes the population of susceptible, infective, and recovered individuals. So, this population should remain nonnegative and bounded.

The following theorem proves that there is a unique globally positive solution of system (2) for any initial value \(X_0 = (S(0), I(0), R(0)) \in \mathbb{R}^3_+\), where

\[
\mathbb{R}^3_+ = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_i > 0, \ i = 1, 2, 3\}.
\]

First, we have the following result.

**Theorem 2.1.** For any given initial value \(X_0 \in \mathbb{R}^3_+\), there exists a unique solution \(X(t) = (S(t), I(t), R(t))\) of system (2) defined on \([0, +\infty)\) and this solution will remain in \(\mathbb{R}^3_+\) with probability one.

**Proof.** From [12], we deduce that system (2) with initial value \(X_0 \in \mathbb{R}^3_+\) admits a unique local solution \(X(t) = (S(t), I(t), R(t))\) on \([0, \tau_e)\), where \(\tau_e\) is the explosion time. To prove that this solution is global, it suffices to prove that \(\tau_e = \infty\) almost surely (briefly a.s.). For this, we define the stopping time \(\tau^+\) by

\[
\tau^+ = \inf\{t \in [0, \tau_e) : S(t) \leq 0 \text{ or } I(t) \leq 0 \text{ or } R(t) \leq 0\},
\]
Letting $\tau = \infty$, where $\emptyset$ denotes the empty set. Clearly, $\tau^+ \leq \tau_e$. Now, we only need to show that $\tau^+ = \infty$ almost surely. Assume that this statement is false, then there exists a constant $T > 0$ such that $\mathbb{P}(\tau^+ < T) > 0$. Define a $C^2$-function $V_1 : \mathbb{R}^3_+ \to \mathbb{R}$ by

$$V_1(X) = \ln(SIR).$$

By Itô’s formula, for for all $t \in [0, \tau^+)$, we obtain

$$dV_1(X(t)) = \left[\frac{A}{S} - \mu_1 - \frac{\beta I}{\psi(S,I)} + \frac{\gamma R}{S} + \frac{\beta S}{\psi(S,I)} - (\mu_2 + r) + r \frac{I}{R} - (\mu_3 + \gamma)\right] dt$$

$$- \frac{1}{2} \left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2 \frac{I^2}{\psi^2(S,I)} + \sigma_4^2 \frac{S^2}{\psi^2(S,I)}\right] dt$$

$$- \sigma_1 dB_1 - \sigma_2 dB_2 - \sigma_3 dB_3 - \sigma_4 \frac{I}{\psi(S,I)} dB_4 + \sigma_4 \frac{S}{\psi(S,I)} dB_4.$$

Since $X(t)$ is positive for all $t \in [0, \tau^+]$ and $\psi(S,I) \geq 1$, we have

$$dV_1(X(t)) \geq f(S,I) dt - \sigma_1 dB_1 - \sigma_2 dB_2 - \sigma_4 \frac{(I - S)}{\psi(S,I)} dB_4 - \sigma_3 dB_3,$$

where $f(S,I) = -\mu_1 - \beta I - \mu_2 - r - \mu_3 - \gamma - \frac{1}{2}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2 I^2 + \sigma_4^2 S^2)$. Hence,

$$V_1(X(t)) \geq V_1(X_0) + \int_0^t f(S(u),I(u)) du - \sigma_1 B_1(t) - \sigma_2 B_2(t) - \sigma_3 B_3(t) - \sigma_4 \int_0^t \frac{I(u) - S(u)}{\psi(S(u),I(u))} dB_4(u).$$

Noticing some components of $X(\tau^+)$ equal 0. Then

$$\lim_{t \to \tau^+} V_1(X(t)) = -\infty.$$

Letting $t \to \tau^+$ in (3) leads to the contradiction

$$-\infty \geq V_1(X_0) + \int_0^{\tau^+} f(S(u),I(u)) du - \sigma_1 B_1(\tau^+) - \sigma_2 B_2(\tau^+) - \sigma_3 B_3(\tau^+)$$

$$- \sigma_4 \int_0^{\tau^+} \frac{I(u) - S(u)}{\psi(S(u),I(u))} dB_4(u) > -\infty.$$

Therefore, $\tau^+ = \infty$ a.s., which means that $X(t) \in \mathbb{R}^3_+$ a.s. for all $t \geq 0$. This completes the proof of the theorem.  

Moreover, it is clear that region

$$\Gamma = \{(S,I,R) \in \mathbb{R}^3_+ : S + I + R \leq \frac{A}{\mu_1}\}$$
is a positive invariant set of the deterministic model (1). Here, we will show that the region $\Gamma$ is almost surely positive invariant set of the corresponding stochastic model (2), i.e., if $X_0 = (S(0), I(0), R(0)) \in \Gamma$, then $\mathbb{P}(X(t) \in \Gamma) = 1$ for all $t \geq 0$.

**Theorem 2.2.** The region $\Gamma$ is almost surely positive invariant of our stochastic model (2).

**Proof.** Let $X_0 \in \Gamma$ and $n_0 > 0$ be sufficiently large such that each component of $X_0$ is contained within the interval $(\frac{1}{n_0}, \frac{A}{\mu_1}]$. Define, for each integer $n \geq n_0$, the stopping times

$$\tau_n = \inf \{ t > 0 : X(t) \in \Gamma \text{ and } X(t) \notin \left( \frac{1}{n}, \frac{A}{\mu_1} \right)^3 \}, \quad \tau = \inf \{ t > 0 : X(t) \notin \Gamma \}.$$ 

We need to prove that $\mathbb{P}(\tau = \infty) = 1$, that is, $\mathbb{P}(\tau < t) = 0$ for all $t > 0$. Obviously, $\mathbb{P}(\tau < t) \leq \mathbb{P}(\tau_n < t)$. So, it suffices to show that $\limsup_{n \to \infty} \mathbb{P}(\tau_n < t) = 0$. For this, we consider a $C^2$-function $V_2 : \mathbb{R}^3_+ \to \mathbb{R}_+$ defined by

$$V_2(X) = \frac{1}{S} + \frac{1}{I} + \frac{1}{R}.$$ 

Applying the Itô formula, for all $t \geq 0$ and $s \in [0, t \wedge \tau_n]$, we get

$$dV_2(X(s)) = \left( -\frac{A}{S^2} + \frac{\mu_1}{S} + \frac{\beta I}{S\psi(S,I)} - \frac{\gamma R}{S} + \frac{\sigma_2^2}{S} + \frac{\sigma_4^2 I^2}{S\psi^2(S,I)} \right) ds$$

$$+ \left( \frac{\mu_2 + r}{I} - \frac{\beta S}{I\psi(S,I)} + \frac{\sigma_2^2}{I} + \frac{\sigma_4^2 S^2}{I\psi^2(S,I)} \right) ds$$

$$+ \left( \frac{\mu_3 + \gamma \sigma_3}{R} - \frac{r I}{R^2} \right) ds + \frac{\sigma_3}{R} dB_3$$

$$\leq \left( \mu_1 + \beta I + \frac{\sigma_2^2}{S} + \frac{\sigma_3^2 I^2}{I} \right) \frac{ds}{S} + \left( \mu_2 + r + \sigma_2^2 + \frac{\sigma_4^2 S^2}{I} \right) \frac{ds}{I}$$

$$+ \left( \mu_3 + \gamma \sigma_3^2 \right) \frac{ds}{R} + \frac{\sigma_1}{S} dB_1 + \frac{\sigma_2}{I} dB_2 + \frac{\sigma_3}{R} dB_3 + \frac{\sigma_4 (I^2 - S^2)}{SI\psi(S,I)} dB_4.$$ 

Hence,

$$dV_2(X(s)) \leq \eta V_2(X(s)) ds + \frac{\sigma_1}{S} dB_1 + \frac{\sigma_2}{I} dB_2 + \frac{\sigma_3}{R} dB_3 + \frac{\sigma_4 (I^2 - S^2)}{SI\psi(S,I)} dB_4, \quad (4)$$

where

$$\eta = \max \left\{ \frac{\mu_1 + \beta A}{\mu_1} + \sigma_1^2 + \frac{\sigma_2^2 A^2}{\mu_1}, \mu_2 + r + \sigma_2^2 + \frac{\sigma_2^2 A^2}{\mu_1}, \mu_3 + \gamma + \sigma_3^2 \right\}.$$
Taking integral and expectations on both sides of (4) and applying Fubini’s theorem, we get
\[
EV_2(X(s)) \leq V_2(X_0) + \eta \int_0^s EV_2(X(u))du.
\]
According to Gronwall inequality, we deduce that
\[
\forall s \in [0, t \wedge \tau_n], \ EV_2(X(s)) \leq V_2(X_0)e^{\eta s}.
\]
Thus
\[
EV_2(X(t \wedge \tau_n)) \leq V_2(X_0)e^{\eta (t \wedge \tau_n)} \leq V_2(X_0)e^{\eta t}, \ \forall t \geq 0. \tag{5}
\]
Since \( V_2(X(t \wedge \tau_n)) > 0 \) and some component of \( X(\tau_n) \) is less than or equal to \( \frac{1}{n} \), we deduce that
\[
EV_2(X(t \wedge \tau_n)) \geq E[V_2(X(\tau_n))\chi\{\tau_n < t\}] \geq n\mathbb{P}(\tau_n < t), \tag{6}
\]
From the inequalities (5) and (6), we get for all \( t \geq 0 \)
\[
\mathbb{P}(\tau_n < t) \leq \frac{V_2(X_0)e^{\eta t}}{n}.
\]
Therefore, \( \limsup_{n \to \infty} \mathbb{P}(\tau_n < t) = 0 \), and the proof is complete.

3 Extinction of the disease

It is easy to see that the basic reproduction number of (1) is given by
\[
R_0^D = \frac{\beta A}{(\mu_1 + \alpha_1 A)(\mu_2 + r)}.
\]
We know the value of the deterministic threshold \( R_0^D \) characterizes the dynamical behaviors of system (1) and guarantees persistence or extinction of the disease. Similarly, we define the following threshold of our stochastic SIRS epidemic model (2) as follows
\[
R_0^S = R_0^D \frac{\mu_2 + r}{\mu_2 + r + \frac{1}{2}\sigma_2^2} \left( 1 - \frac{\sigma_4^2 A}{2\beta(\mu_1 + \alpha_1 A)} \right).
\]
Obviously, \( R_0^S = R_0^D \) when \( \sigma_2 = \sigma_4 = 0 \).
Theorem 3.1. Let \( X(t) = (S(t), I(t), R(t)) \) be the solution of system (2) with initial value \( X(0) \in \Gamma \). Assume that (a) \( \sigma_4^2 > \frac{\beta^2}{2(\mu_2 + r + \frac{1}{2} \sigma_2^2)} \), or (b) \( R_0^S < 1 \) and \( \sigma_4^2 \leq \frac{\beta(\mu_1 + \alpha_A)}{A} \). Then

\[
\begin{align*}
\limsup_{t \to \infty} \frac{\ln I(t)}{t} &\leq \frac{\beta^2}{2\sigma_4^2} - (\mu_2 + r + \frac{1}{2} \sigma_2^2) < 0 \text{ a.s. if (a) holds}, \\
\limsup_{t \to \infty} \frac{\ln I(t)}{t} &\leq (\mu_2 + r + \frac{1}{2} \sigma_2^2)(R_0^S - 1) < 0 \text{ a.s. if (b) holds},
\end{align*}
\]

namely, \( I(t) \) tends to zero exponentially a.s., i.e., the disease dies out with probability 1.

Proof. It follows from Itô’s formula that

\[
d\ln I = \left[ \frac{\beta S}{\psi(S, I)} - (\mu_2 + r) - \frac{1}{2} \left( \sigma_2^2 + \sigma_4^2 \frac{S^2}{\psi^2(S, I)} \right) \right] dt - \sigma_2 dB_2 + \sigma_4 \frac{S}{\psi(S, I)} dB_4.
\]

Integrating this from 0 to \( t \) and dividing by \( t \) on both sides, we have

\[
\frac{\ln I(t)}{t} = \frac{1}{t} \int_0^t \left[ \frac{\beta S(u)}{\psi(S, I)} - (\mu_2 + r) - \frac{1}{2} \left( \sigma_2^2 + \sigma_4^2 \frac{S^2(u)}{\psi^2(S, I)} \right) \right] du \\
+ \frac{\ln I(0)}{t} - \sigma_2 \frac{B_2(t)}{t} + \frac{M(t)}{t},
\]

where \( M(t) = \int_0^t \sigma_4 \frac{S(u)}{\psi(S(u), I(u))} dB_4(u) \). By the large number theorem for martingales (see e.g. Ref. [12]), we have

\[
\lim_{t \to \infty} \frac{M(t)}{t} = \lim_{t \to \infty} \frac{B_1(t)}{t} = 0 \text{ a.s.}
\]

If the condition (a) is satisfied, Eq. (10) becomes

\[
\frac{\ln I(t)}{t} = \frac{1}{t} \int_0^t \left[ -\frac{1}{2} \sigma_4^2 \frac{S(u)}{\psi(S, I)} - \frac{\beta}{\sigma_2^2} \right] du \\
+ \frac{\ln I(0)}{t} - \sigma_2 \frac{B_2(t)}{t} + \frac{M(t)}{t} \\
\leq \left[ -\left( \mu_2 + r + \frac{1}{2} \sigma_2^2 \right) + \frac{\beta^2}{2\sigma_4^2} \right] + \frac{\ln I(0)}{t} - \sigma_2 \frac{B_2(t)}{t} + \frac{M(t)}{t}.
\]

Taking the limit superior of both sides, we obtain the desired assertion (8).

If (b) holds, then

\[
\frac{\ln I(t)}{t} \leq \left[ \frac{\beta A}{\mu_1 + \alpha_A} - (\mu_2 + r + \frac{1}{2} \sigma_2^2) - \frac{\sigma_4^2 A^2}{2(\mu_1 + \alpha_A)^2} \right] + \frac{\ln I(0)}{t} - \sigma_2 \frac{B_2(t)}{t} + \frac{M(t)}{t} \\
\leq (R_0^S - 1)(\mu_2 + r + \frac{1}{2} \sigma_2^2) + \frac{\ln I(0)}{t} - \sigma_2 \frac{B_2(t)}{t} + \frac{M(t)}{t}
\]

(11)
Taking the limit superior of both sides, we get the assertion (9). We have proved that
\[
\limsup_{t \to \infty} \frac{\ln I(t)}{t} \leq \lambda_I < 0 \ a.s.,
\] (12)
where \( \lambda_I = \frac{\beta^2}{2\sigma^2} - (\mu_2 + r + \frac{1}{2}\sigma_2^2) \) if (a) holds, and \( \lambda_I = (\mu_2 + r + \frac{1}{2}\sigma_2^2)(R_0^S - 1) \) if (b) holds. This finishes the proof. ■

4 Persistence of the disease

In this section, we investigate the conditions for the persistence of the disease. For simplicity, we introduce the following notation:
\[
\langle x(t) \rangle = \frac{1}{t} \int_0^t x(s)ds.
\]
At first, we recall the definition of persistence in the mean.

**Definition 4.1.** System (2) is said to be persistent in the mean, if
\[
\liminf_{t \to \infty} \langle I(t) \rangle > 0 \ a.s.
\]
Further, we need the following lemma (see Lemma 5.1. in [5]).

**Lemma 4.2.** Let \( g \in C([0, \infty) \times \Omega, (0, \infty)) \) and \( G \in C([0, \infty) \times \Omega, \mathbb{R}) \). If there exist positive constants \( \lambda_0, \lambda \) such that
\[
\ln g(t) \geq \lambda_0 t - \lambda \int_0^t g(s)ds + G(t) \ a.s.
\]
for all \( t \geq 0 \), and \( \lim_{t \to \infty} \frac{G(t)}{t} = 0 \ a.s. \), then
\[
\liminf_{t \to \infty} \langle g(t) \rangle \geq \frac{\lambda_0}{\lambda} \ a.s.
\]

**Theorem 4.3.** If \( R_0^S > 1 \), then the solution \((S(t), I(t), R(t))\) of system (2) with initial initial value \( X_0 \in \Gamma \) is persistent in mean. Moreover, we have
\[
\liminf_{t \to \infty} \langle I(t) \rangle \geq I_* > 0,
\] (13)
\[
\liminf_{t \to \infty} \langle R(t) \rangle \geq \frac{rI_*}{\mu_3 + \gamma} > 0,
\] (14)
\[
\liminf_{t \to \infty} \left( \frac{A}{\mu_1} - S(t) \right) \geq \frac{[\mu_2(\mu_3 + \gamma) + \mu_3r]I_*}{\mu_1(\mu_3 + \gamma)} > 0,
\] (15)
where
\[
I_* = \frac{(R_0^S - 1)(\mu_2 + r + \frac{1}{2}\sigma_2^2)\beta^{-1} \mu_1(\gamma + \mu_3)(\mu_1 + \alpha_1 A)}{\gamma \mu_1 \mu_2 + \beta A \mu_3 + A(\alpha_2 \mu_1 + \alpha_3 A)(\gamma + \mu_3)}.
\]
Proof. We consider a function $W$ defined in $\Gamma$ by

$$W(X) = \omega_1(S + I + R) + \omega_2S + \ln I.$$  

It follows from Itô’s formula that

$$dW = \omega_1\left( A - \mu_1S - \mu_2I - \mu_3R \right)dt - \omega_1\sigma_1SdB_1 - \omega_1\sigma_2IdB_2 - \omega_1\sigma_3RdB_3$$
$$+ \omega_2\left( A - \mu_1S - \frac{\beta SI}{\psi(S, I)} + \gamma R \right)dt - \omega_2\sigma_1SdB_1 - \omega_2\sigma_4I\frac{dSI}{\psi(S, I)}dB_4$$
$$+ \left(- (\mu_2 + r) - \frac{1}{2}\sigma_2^2 - \frac{1}{2}\left(\frac{\sigma_2^2}{\psi(S, I)^2} + \frac{\beta S}{\psi(S, I)}\right) \right)dt$$
$$- \sigma_2dB_2 + \frac{\sigma_4SI}{\psi(S, I)}dB_4.$$  

Since $X \in \Gamma$, we have

$$-\frac{\beta SI}{\psi(S, I)} \geq -\frac{\beta A}{\mu_1}$$  

and

$$\frac{\beta S}{\psi(S, I)} = \frac{\beta A}{\mu_1 + \alpha_1A} - \frac{\beta \mu_1}{\mu_1 + \alpha_1A} \left( \frac{A}{\mu_1} - S \right) - \frac{\beta A}{\mu_1 + \alpha_1A} \left( \frac{\alpha_2}{A} + \frac{A}{\mu_1} \right)I.$$  

Substituting (17) and (18) into (16), we get

$$dW \geq (R_S - 1)(\mu_2 + r + \frac{1}{2}\sigma_2^2)dt - (\omega_1\mu_2 + \omega_2\frac{\beta A}{\mu_1} + \frac{\beta A}{\mu_1 + \alpha_1A}(\alpha_2 + \frac{A}{\mu_1}))(\omega_1 + \omega_2)\mu_1$$
$$+ ((\omega_1 + \omega_2)\mu_1 - \frac{\beta \mu_1}{\mu_1 + \alpha_1A})(\frac{A}{\mu_1} - S)dt + (\omega_2\gamma - \omega_1\mu_3)Rdt$$
$$- (\omega_1 + \omega_2)\sigma_1SdB_1 - (1 + \omega_1I)\sigma_2dB_2 - \omega_1\sigma_3RdB_3$$
$$- (\omega_2I - 1)\sigma_4\frac{S}{\psi(S, I)}dB_4.$$  

In order to eliminate $\frac{A}{\mu_1} - S$ and $R$ from (19), we choose $\omega_1$ and $\omega_2$ as follows

$$\omega_1 = \frac{\beta \gamma}{(\gamma + \mu_3)(\mu_1 + \alpha_1A)} \text{ and } \omega_2 = \frac{\beta \mu_3}{(\gamma + \mu_3)(\mu_1 + \alpha_1A)}$$  

Substituting this in (19) and integrating we get

$$W(X(t)) \geq W(X_0) + (R_S - 1)(\mu_2 + r + \frac{1}{2}\sigma_2^2)t - \gamma \mu_2 + \frac{\beta A}{\mu_1} + \frac{\beta A}{\mu_1 + \alpha_1A}(\alpha_2 + \frac{A}{\mu_1})(\gamma + \mu_3)\int_0^t Ids$$
$$- \int_0^t (\omega_1 + \omega_2)\sigma_1SdB_1 - \int_0^t (1 + \omega_1I)\sigma_2dB_2$$
$$- \int_0^t \omega_1\sigma_3RdB_3 - \int_0^t (\omega_2I - 1)\sigma_4\frac{S}{\psi(S, I)}dB_4.$$
Hence,
\[
\ln I \geq (R_S - 1)(\mu_2 + r + \frac{1}{2}\sigma_2^2)t - \frac{\gamma \mu_1 \mu_2 + \beta A \mu_3 + A(\alpha_2 \mu_1 + \alpha_3 A)(\gamma + \mu_3)}{\beta - 1} \frac{\gamma \mu_1 (\gamma + \mu_3)(\mu_1 + \alpha_1 A)}{t - \gamma} \int_0^t Ids + H(t),
\]
where
\[
H(t) = W(X_0) - (\omega_1 + \omega_2)S - \omega_1 I - \omega_1 R - \int_0^t (\omega_1 + \omega_2)\sigma_1 S dB_1 - \int_0^t (1 + \omega_1)\sigma_2 dB_2 \nonumber \\
- \int_0^t \omega_1 \sigma_3 RdB_3 - \int_0^t (\omega_2 I - 1)\frac{\sigma_4 S}{\psi(S, I)} dB_4.
\]
From the large number theorem for martingales, we deduce that
\[
\lim_{t \to \infty} \frac{H(t)}{t} = 0 \ a.s.
\]
By using Lemma 4.2, we obtain the first desired assertion
Next, the third equation of system (2) gives
\[
\langle R(t) \rangle = \frac{r}{(\mu_3 + \gamma)}(I(t)) + \tilde{H}(t).
\]
where
\[
\tilde{H}(t) = \frac{R_0 - R}{(\mu_3 + \gamma)t} - \frac{\sigma_3}{(\mu_3 + \gamma)t} \int_0^t RdB_3.
\]
We have \( \lim_{t \to \infty} \tilde{H}(t) = 0 \ a.s. \) By (21) and (13), we get the inequality (14).

Finally, we prove the assertion (15). We have
\[
d(S + I + R) = \left( \mu_1(\frac{A}{\mu_1} - S) - \mu_2 I - \mu_3 R \right) dt - \sigma_1 S dB_1 - \sigma_2 IdB_2 - \sigma_3 RdB_3.
\]
Therefore,
\[
\langle \frac{A}{\mu_1} - S(t) \rangle = \frac{\mu_2}{\mu_1} \langle I(t) \rangle + \frac{\mu_3}{\mu_1} \langle R(t) \rangle + \tilde{H}(t).
\]
where \( \tilde{H}(t) = \frac{N - N_0}{\mu_1 t} + \frac{1}{\mu_1} \int_0^t \sigma_1 S dB_1 + \frac{1}{\mu_1} \int_0^t \sigma_2 IdB_2 + \frac{1}{\mu_1} \int_0^t \sigma_3 RdB_3. \) Since \( \lim_{t \to \infty} \tilde{H}(t) = 0 \ a.s., \) and from (13) and (14) we get the last inequality (15).

5 Numerical simulations

In this section, we present some numerical simulations to illustrate our main results.
Example 5.1. i) First, we choose the parameter values of our stochastic system (2) as follows: $A = 4$, $\mu_1 = 0.01$, $\mu_2 = 0.02$, $\mu_3 = 0.01$, $\beta = 0.005$, $r = 0.01$, $\gamma = 0.2$, $\alpha_1 = 0.1$, $\alpha_2 = 0.1$, $\alpha_3 = 1$, $\sigma_1 = 0.002$, $\sigma_2 = 0.015$, $\sigma_3 = 0.04$ and $\sigma_4 = 0.04$. By a simple calculation, we have $\sigma_4^2 = 0.0016 > \frac{\beta^2}{2(\mu_2 + r + \frac{1}{2}\sigma_2^2)} = 0.00047619$. Hence, the condition (a) of Theorem 3.1 is satisfied. Therefore, the disease dies out.

ii) Next, we choose $\sigma_4 = 0.02$ and we keep the other parameter values. In this case we have $\sigma_4^2 = 0.0004 < \frac{\beta^2}{2(\mu_2 + r + \frac{1}{2}\sigma_2^2)} = 0.0005125$ and the basic reproduction number of our stochastic model (2) satisfies $R_0^S = 0.72107 < 1$, which implies that the condition (b) of Theorem 3.1 is verified. Then the disease dies out. Figure 1 illustrates the both cases i and ii.

![Figure 1: Trajectories of I(t) where the one on the left represents the case (i) of Example 5.1 and the second the case(ii).](image)

Example 5.2. To check the results obtained in Theorem 4.3, we choose the following parameters: $A = 4$, $\mu_1 = 0.01$, $\mu_2 = 0.002$, $\mu_3 = 0.001$, $\beta = 0.09$, $r = 0.01$, $\gamma = 0.02$, $\alpha_1 = 0.1$, $\alpha_2 = 0.1$, $\alpha_3 = 0.1$, $\sigma_1 = 0.001$, $\sigma_2 = 0.005$, $\sigma_3 = 0.002$ and $\sigma_4 = 0.04$. For this set of parameters we have $R_0^S = 7.8608 > 1$. Therefore, according to Theorem 4.3, the solution of system (2) is persistent in mean. This result is illustrated in Figure 2.

6 Discussion and conclusion

In this work, we have proposed and analyzed a new stochastic SIRS epidemic model with specific functional response by introducing the noise in the mortality rates and infection coefficient. The specific functional response modeled the transmission rate of disease and it covers various types of incidence rate existing in the literature such as the mass action called also the bilinear incidence rate; the saturation incidence; Beddington-DeAngelis response and Crowley-Martin response. Furthermore, the stochastic SIRS epidemic models presented
in [15, 11, 4, 6, 16, 8, 14, 7] are extended and generalized. Firstly, we have proved the global existence, positivity and boundedness of solutions.

In addition, we have proved that the dynamics of our stochastic model are fully determined by the threshold parameter $R^S_0$ when the intensity of white noise is small. More precisely, we have proved that the disease dies out if $R^S_0 < 1$ and the intensity of white noise is not large (see Theorem 3.1). Whereas the disease persists in the population when $R^S_0 > 1$ (see Theorem 4.3). On the other hand, in absence of noise, the threshold $R^S_0$ is equal to the basic reproduction number of the corresponding deterministic model. For these reasons, we conclude that when the intensity of white noise is small, the threshold $R^S_0$ can be considered as the basic reproduction number of our stochastic model that guarantees the persistence or extinction of the disease.

References


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