A Novel Generalization of Fuzzy Ideals in Semigroups

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Abstract

The notion of \((\tilde{\alpha}, \tilde{\beta})\)-fuzzy left (right, bi-) ideals in semigroups is introduced, and related properties are investigated. Characterizations of \((\in, \in \lor q_0)\)-fuzzy left (right, bi-) ideals are provided.

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1 Introduction

The idea of quasicoincidence of a fuzzy point with a fuzzy set, which is mentioned in [10], played a vital role to generate some different types of fuzzy subgroups. It is worth pointing out that Bhakat and Das [2, 3] gave the concepts of \((\alpha, \beta)\)-fuzzy subgroups by using the “belongs to” relation \((\in)\) and “quasicoincident with” relation \((q)\) between a fuzzy point and a fuzzy subgroup, and
introduced the concept of an \((\varepsilon, \in \vee q)\)-fuzzy subgroup. In particular, \((\varepsilon, \in \vee q)\)-fuzzy subgroup is an important and useful generalization of Rosenfeld’s fuzzy subgroup. Bhakat et al. applied the \((\varepsilon, \in \vee q)\)-fuzzy type to group and ring (see [1, 2, 3, 4]). Dudek et al. [5] characterized different types of \((\alpha, \beta)\)-fuzzy ideals of hemirings. In [8] Jun and Song initiated the study of \((\alpha, \beta)\)-fuzzy interior ideals of a semigroup. In [9] Kazanci and Yamak study \((\varepsilon, \in \vee q)\)-fuzzy bi-ideals of a semigroup. Shabir et al. [11] introduced the concept of \((\alpha, \beta)\)-fuzzy ideal, \((\alpha, \beta)\)-fuzzy generalized bi-ideal, and characterized regular semigroups by the properties of these ideals. Jun et al. [6] considered more general form of quasi-coincident fuzzy point, and they [7] introduced the notions of \((\tilde{\alpha}, \tilde{\beta})\)-fuzzy subsemigroups in semigroups, and investigate related properties. They provided characterizations of \((\varepsilon, \in \vee q_0^\delta)\)-fuzzy subsemigroups, and considered a condition for an \((\varepsilon, \in \vee q_0^\delta)\)-fuzzy subsemigroup to be an \((\varepsilon, \varepsilon)\)-fuzzy subsemigroup. Given a fuzzy set with finite images, they established an \((\varepsilon, \in \vee q_0^\delta)\)-fuzzy subsemigroup generated by the given fuzzy set. Yuan et al. [12] provided a generalization of fuzzy subgroups and \((\varepsilon, \in \vee q_0^\delta)\)-fuzzy subgroups.

The aim of this paper is to generalize the notions and results in the paper [11]. We introduce the notions of \((\tilde{\alpha}, \tilde{\beta})\)-fuzzy left (right, bi-) ideals in semigroups, and investigate related properties. We discuss characterizations of \((\varepsilon, \in \vee q_0^\delta)\)-fuzzy left (right, bi-) ideals.

\section{Preliminaries}

An element \(a\) of a semigroup \(S\) is called a regular element if there exists an element \(x\) of \(S\) such that \(a = axa\). A semigroup \(S\) is said to be regular if every element of \(S\) is regular.

A nonempty subset \(B\) of a semigroup \(S\) is called

- a subsemigroup of \(S\) if \(B^2 \subseteq B\),
- a left (resp. right) ideal of \(S\) if \(SB \subseteq B\) (resp. \(BS \subseteq B\)),
- a generalized bi-ideal of \(S\) if \(BSB \subseteq B\),
- a bi-ideal of \(S\) if it is both subsemigroup and a generalized bi-ideal of \(S\).

For two fuzzy set \(\lambda\) and \(\nu\) in \(S\), we say \(\lambda \leq \nu\) if \(\lambda(x) \leq \nu(x)\) for all \(x \in S\). We define \(\lambda \land \nu\) and \(\lambda \lor \nu\) as follows:

\[ \lambda \land \nu : S \to [0, 1], \ x \mapsto \min\{\lambda(x), \nu(x)\} \]

and

\[ \lambda \lor \nu : S \to [0, 1], \ x \mapsto \max\{\lambda(x), \nu(x)\} \]
respectively. The product $\lambda \circ \nu$ of $\lambda$ and $\nu$ is defined to be fuzzy set in $S$ as follows:

$$\lambda \circ \nu(x) := \begin{cases} \bigvee_{x=yz} \min\{\lambda(y), \nu(z)\} & \text{if } \exists y, z \in S \text{ such that } x = yz, \\ 0 & \text{otherwise.} \end{cases}$$

A fuzzy set $\lambda$ in a set $S$ of the form

$$\lambda(y) := \begin{cases} t \in (0, 1] & \text{if } y = x, \\ 0 & \text{if } y \neq x, \end{cases}$$

is said to be a fuzzy point with support $x$ and value $t$ and is denoted by $x_t$. It is clear that $x_t \circ y_r = (xy)_{\min\{t, r\}}$ for all fuzzy points $x_t$ and $y_r$ in a set $S$.

For a fuzzy point $x_t$ and a fuzzy set $\lambda$ in a set $S$, we say that

- $x_t \in \lambda$ (resp. $x_t \in q_0 \lambda$) (see [10]) if $\lambda(x) \geq t$ (resp. $\lambda(x) + t > 1$). In this case, $x_t$ is said to belong to (resp. be quasi-coincident with) a fuzzy set $\lambda$.

- $x_t \in q_0 \lambda$ (resp. $x_t \in \wedge q_0 \lambda$) (see [10]) if $x_t \in \lambda$ or $x_t q_0 \lambda$ (resp. $x_t \in \lambda$ and $x_t q_0 \lambda$).

Let $\delta \in (0, 1]$. For a fuzzy point $x_t$ and a fuzzy set $\lambda$ in a set $X$, we say that

- $x_t$ is a $\delta$-quasi-coincident with $\lambda$, written $x_t q_0^\delta \lambda$, (see [6]) if $\lambda(x) + t > \delta$,

- $x_t \in q_0^\delta \lambda$ (resp. $x_t \in \wedge q_0^\delta \lambda$) (see [6]) if $x_t \in \lambda$ or $x_t q_0^\delta \lambda$ (resp. $x_t \in \lambda$ and $x_t q_0^\delta \lambda$).

Obviously, $x_t q_0 \lambda$ implies $x_t q_0^\delta \lambda$. If $\delta = 1$, then the $\delta$-quasi-coincident with $\lambda$ is the quasi-coincident with $\lambda$, that is, $x_t q_0^1 \lambda = x_t q_0 \lambda$.

For $\alpha \in \{\in, q, \in \vee q, \in \wedge q, \in \vee q_0^\delta, \in \wedge q_0^\delta\}$, we say that $x_t \alpha \lambda$ if $x_t \alpha \lambda$ does not hold.

### 3 Generalized fuzzy left (right, bi-) ideals

In what follows, let $\delta$ be an element of $(0, 1]$ and let $S$ be a semigroup and $\tilde{\alpha}$ and $\tilde{\beta}$ denote any one of $\in$, $q_0^\delta$, $\in \vee q_0^\delta$ and $\in \wedge q_0^\delta$ unless otherwise specified.

**Definition 3.1 ([7])** A fuzzy set $\lambda$ in $S$ is called an $(\tilde{\alpha}, \tilde{\beta})$-fuzzy subsemigroup of $S$, where $\tilde{\alpha} \neq \in \wedge q_0^\delta$, if

$$\left( \forall x, y \in S \right) \left( \forall t, r \in (0, \delta] \right) \left( x_t \tilde{\alpha} \lambda, y_r \tilde{\alpha} \lambda \Rightarrow x_t \circ y_r \tilde{\beta} \lambda \right).$$

(2)
Let $\lambda$ be a fuzzy set in $S$ such that $\lambda(x) \leq \frac{\delta}{2}$ for all $x \in S$. Let $x \in S$ and $t \in (0, \delta]$ be such that $x_t \in \land q^0_0 \lambda$. Then $\lambda(x) \geq t$ and $\lambda(x) + t > \delta$. It follows that $\delta < \lambda(x) + t \leq 2\lambda(x)$, so that $\lambda(x) \geq \frac{\delta}{2}$. This means that $\{x_t \mid x_t \in \land q^0_0 \lambda\} = \emptyset$. Hence the case $\tilde{\alpha} = \land q^0_0$ should be omitted.

**Lemma 3.2 ([7])** A fuzzy set $\lambda$ in $S$ is an $(\in, \in \lor q^0_0)$-fuzzy subsemigroup of $S$ if and only if

$$\forall x, y \in S \left( \lambda(xy) \geq \min\{\lambda(x), \lambda(y), \frac{\delta}{2}\} \right).$$

(3)

**Definition 3.3** A fuzzy set $\lambda$ in $S$ is called an $(\tilde{\alpha}, \tilde{\beta})$-fuzzy left (resp. right) ideal of $S$ if for any $x, y \in S$ and $t \in (0, \delta]$, $y_t \tilde{\alpha} \lambda \Rightarrow (xy)_t \tilde{\beta} \lambda$ (resp. $(yx)_t \tilde{\beta} \lambda$). (4)

A fuzzy set $\lambda$ in $S$ is called an $(\tilde{\alpha}, \tilde{\beta})$-fuzzy ideal of $S$ if it is both an $(\tilde{\alpha}, \tilde{\beta})$-fuzzy left ideal and an $(\tilde{\alpha}, \tilde{\beta})$-fuzzy right ideal of $S$.

**Definition 3.4** A fuzzy set $\lambda$ in $S$ is called an $(\tilde{\alpha}, \tilde{\beta})$-fuzzy bi-ideal of $S$ if it satisfies the condition (2) and for any $x, y, z \in S$ and $t, r \in (0, \delta]$, $x_t \tilde{\alpha} \lambda, z_t \tilde{\alpha} \lambda \Rightarrow (xyz)_{\min(t,r)} \tilde{\beta} \lambda$. (5)

If a fuzzy set $\lambda$ in $S$ satisfies the condition (5) only is called an $(\tilde{\alpha}, \tilde{\beta})$-fuzzy generalized bi-ideal of $S$.

**Example 3.5** Consider a semigroup $S = \{a, b, c, d\}$ with the multiplication $\cdot$ which is described by Table 1.

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Let $\lambda$ be a fuzzy set in $S$ defined by

$$\lambda : S \to [0, 1], \ x \mapsto \begin{cases} \frac{\delta}{2} & \text{if } x = a, \\ \frac{\delta}{5} & \text{if } x = c, \\ \frac{\delta}{10} & \text{if } x \in \{b, d\}, \end{cases}$$

Then $\lambda$ is an $(\in, \in \lor q^0_0)$-fuzzy generalized bi-ideal of $S$. 
Theorem 3.6 For a subset \( Q \) of \( S \), consider a fuzzy set \( \lambda \) in \( S \) as follows:

\[
\lambda : S \rightarrow [0, 1], \quad x \mapsto \begin{cases} 
\varepsilon & \text{if } x \in Q, \\
0 & \text{if } x \notin Q,
\end{cases}
\]

where \( \varepsilon \geq \frac{\delta}{2} \). If \( Q \) is a left (resp., right) ideal of \( S \), then \( \lambda \) is both an \((\in, \in \lor \delta, 0)\)-fuzzy left (resp., right) ideal and a \((q^0, \in \lor q^0)\)-fuzzy left (resp., right) ideal of \( S \).

Proof. Assume that \( Q \) is a left ideal of \( S \). Let \( x, y \in S \) and \( t \in (0, \delta] \) such that \( y_t \in \lambda \). Then \( \lambda(y) \geq t > 0 \), and so \( \lambda(y) = \varepsilon \), i.e., \( y \in Q \). Hence \( xy \in Q \), and thus \( \lambda(xy) = \varepsilon \). If \( t \leq \frac{\delta}{2} \), then \( \lambda(xy) + t = \varepsilon + t > \frac{\delta}{2} + \frac{\delta}{2} = \delta \), and so \((xy)_t \in q^0 \lambda \). Therefore \((xy)_t \in \lor q^0 \lambda \), and \( \lambda \) is an \((\in, \in \lor q^0)\)-fuzzy left ideal of \( S \). Now, let \( x, y \in S \) and \( t \in (0, \delta] \) such that \( y_t \in \lambda \). Then \( \lambda(y) + t > \delta \), which imply that \( y \in Q \). Hence \( xy \in Q \) and \( \lambda(xy) = \varepsilon \). If \( t \leq \frac{\delta}{2} \), then \( \lambda(xy) = \varepsilon \geq \frac{\delta}{2} \geq t \), and so \((xy)_t \in \lambda \). If \( t > \frac{\delta}{2} \), then

\[
\lambda(xy) + t = \varepsilon + t > \frac{\delta}{2} + \frac{\delta}{2} = \delta,
\]

and so \((xy)_t \in q^0 \lambda \). Therefore \((xy)_t \in \lor q^0 \lambda \), and \( \lambda \) is a \((q^0, \lor q^0)\)-fuzzy left ideal of \( S \).

Theorem 3.6 is a generalization of Theorem 7 in [11], that is, we have the following corollary.

Corollary 3.7 ([11]) For a left (resp., right) ideal \( Q \) of \( S \), consider a fuzzy set \( \lambda \) in \( S \) as follows:

\[
\lambda : S \rightarrow [0, 1], \quad x \mapsto \begin{cases} 
\varepsilon & \text{if } x \in Q, \\
0 & \text{if } x \notin Q,
\end{cases}
\]

where \( \varepsilon \geq 0.5 \). Then \( \lambda \) is both an \((\in, \lor q)\)-fuzzy left (resp., right) ideal and a \((q, \lor q)\)-fuzzy left (resp., right) ideal of \( S \).

Theorem 3.8 For a subset \( Q \) of \( S \), consider a fuzzy set \( \lambda \) in \( S \) as follows:

\[
\lambda : S \rightarrow [0, 1], \quad x \mapsto \begin{cases} 
\varepsilon & \text{if } x \in Q, \\
0 & \text{if } x \notin Q,
\end{cases}
\]

where \( \varepsilon \geq \frac{\delta}{2} \). If \( Q \) is a \((\text{generalized})\) bi-ideal of \( S \), then \( \lambda \) is both an \((\in, \lor q^0)\)-fuzzy \((\text{generalized})\) bi-ideal and a \((q^0, \lor q^0)\)-fuzzy \((\text{generalized})\) bi-ideal of \( S \).

Proof. It is similar to the proof of Theorem 3.6.
Corollary 3.9 ([11]) For a (generalized) bi-ideal \( Q \) of \( S \), consider a fuzzy set \( \lambda \) in \( S \) as follows:

\[
\lambda : S \to [0, 1], \quad x \mapsto \begin{cases} 
\varepsilon & \text{if } x \in Q, \\
0 & \text{if } x \notin Q,
\end{cases}
\]

where \( \varepsilon \geq 0.5 \). Then \( \lambda \) is both an \((\varepsilon, \in \cup q)\)-fuzzy (generalized) bi-ideal and a \((q, \in \cup q)\)-fuzzy (generalized) bi-ideal of \( S \).

We provide characterizations of \((\varepsilon, \in \cup q)\)-fuzzy left (resp., right) ideals.

Theorem 3.10 For any fuzzy set \( \lambda \) in \( S \), the following are equivalent.

(i) \( \lambda \) is an \((\varepsilon, \in \cup q)\)-fuzzy left (resp., right) ideal of \( S \).

(ii) \((\forall x, y \in S) (\lambda(xy) \geq \min\{\lambda(y), \delta \} \quad \text{(resp. } \lambda(xy) \geq \min\{\lambda(x), \delta \})\).

Proof. Let \( \lambda \) be an \((\varepsilon, \in \cup q)\)-fuzzy left ideal of \( S \). Suppose that \( \lambda(ab) < \min\{\lambda(b), \frac{\delta}{2}\} \) for some \( a, b \in S \). Take \( t \in (0, \delta) \) such that \( \lambda(ab) < t \leq \min\{\lambda(b), \frac{\delta}{2}\} \). Then \( b_1 \in \lambda, (ab)_0 \in \lambda \) and \( \lambda(ab) + t < 2t \leq \delta \). This is a contradiction, and so \( \lambda(xy) \geq \min\{\lambda(y), \frac{\delta}{2}\} \) for all \( x, y \in S \).

Conversely, assume that \( \lambda(xy) \geq \min\{\lambda(y), \frac{\delta}{2}\} \) for all \( x, y \in S \). Let \( y_1 \in \lambda \). Then

\[
\lambda(xy) \geq \min\{\lambda(y), \frac{\delta}{2}\} \geq \min\{t, \frac{\delta}{2}\} = \begin{cases} 
t & \text{if } t \leq \frac{\delta}{2}, \\
\frac{\delta}{2} & \text{if } t > \frac{\delta}{2},
\end{cases}
\]

and so \((xy_1) \in \lambda \) or \( \lambda(xy) + t > \frac{\delta}{2} + \frac{\delta}{2} = \delta \) Hence \((xy_1) \in \in \cup q_0 \lambda \). Therefore \( \lambda \) is an \((\varepsilon, \in \cup q)\)-fuzzy left ideal of \( S \).

Corollary 3.11 For any fuzzy set \( \lambda \) in \( S \), the following are equivalent.

(i) \( \lambda \) is an \((\varepsilon, \in \cup q)\)-fuzzy ideal of \( S \).

(ii) \((\forall x, y \in S) (\lambda(xy) \geq \min\{\lambda(y), \frac{\delta}{2}\} \quad \text{and } \lambda(xy) \geq \min\{\lambda(x), \frac{\delta}{2}\})\).

Lemma 5 in [11] is a special case of Theorem 3.10, that is,

Corollary 3.12 For any fuzzy set \( \lambda \) in \( S \), the following are equivalent.

(i) \( \lambda \) is an \((\varepsilon, \in \cup q)\)-fuzzy left (resp., right) ideal of \( S \).

(ii) \((\forall x, y \in S) (\lambda(xy) \geq \min\{\lambda(y), 0.5\} \quad \text{(resp. } \lambda(xy) \geq \min\{\lambda(x), 0.5\})\).

Theorem 3.13 If \( \lambda \) is an \((\varepsilon, \in \cup q)\)-fuzzy left ideal and \( \nu \) is an \((\varepsilon, \in \cup q)\)-fuzzy right ideal of \( S \), then \( \lambda \circ \nu \) is an \((\varepsilon, \in \cup q)\)-fuzzy ideal of \( S \).
Proof. For any \(x, y \in S\), we have
\[
\min\{(\lambda \circ \nu)(y), \frac{\delta}{2}\} = \min \left\{ \left( \bigvee_{y=ab} \min\{\lambda(a), \nu(b)\} \right), \frac{\delta}{2} \right\}
\]
\[
= \bigvee_{y=ab} \min\{\lambda(a), \nu(b)\}
\]
\[
\leq \bigvee_{y=ab} \min\{\lambda(xa), \nu(b)\}
\]
\[
\leq \bigvee_{xy=pq} \min\{\lambda(p), \nu(q)\}
\]
\[
= (\lambda \circ \nu)(xy).
\]
Similarly, we have \((\lambda \circ \nu)(xy) \geq \min\{(\lambda \circ \nu)(x), \frac{\delta}{2}\}\). It follows from Theorem 3.10 that \(\lambda \circ \nu\) is an \((\epsilon, \epsilon \vee q_0^\delta)\)-fuzzy ideal of \(S\).

Corollary 3.14 ([11]) If \(\lambda\) is an \((\epsilon, \epsilon \vee q)\)-fuzzy left ideal and \(\nu\) is an \((\epsilon, \epsilon \vee q)\)-fuzzy right ideal of \(S\), then \(\lambda \circ \nu\) is an \((\epsilon, \epsilon \vee q)\)-fuzzy ideal of \(S\).

Theorem 3.15 The intersection of \((\epsilon, \epsilon \vee q_0^\delta)\)-fuzzy left (resp., right) ideals is an \((\epsilon, \epsilon \vee q_0^\delta)\)-fuzzy left (resp., right) ideal.

Proof. Let \(\{\lambda_i \mid i \in \Lambda\}\) be a family of \((\epsilon, \epsilon \vee q_0^\delta)\)-fuzzy left ideals of \(S\). For any \(x, y \in S\), we have
\[
\left( \bigwedge_{i \in \Lambda} \lambda_i \right)(xy) = \bigwedge_{i \in \Lambda} (\lambda_i(xy)) \geq \bigwedge_{i \in \Lambda} \min\{\lambda_i(y), \frac{\delta}{2}\}
\]
\[
= \min \left\{ \left( \bigwedge_{i \in \Lambda} \lambda_i(y) \right), \frac{\delta}{2} \right\}
\]
\[
= \min \left\{ \left( \bigwedge_{i \in \Lambda} \lambda_i \right)(y), \frac{\delta}{2} \right\}.
\]
Therefore \(\bigwedge_{i \in \Lambda} \lambda_i\) is an \((\epsilon, \epsilon \vee q_0^\delta)\)-fuzzy left ideal of \(S\).

Corollary 3.16 ([11]) The intersection of \((\epsilon, \epsilon \vee q)\)-fuzzy left (resp., right) ideals is an \((\epsilon, \epsilon \vee q)\)-fuzzy left (resp., right) ideal.

Corollary 3.17 The intersection of \((\epsilon, \epsilon \vee q)\)-fuzzy ideals is an \((\epsilon, \epsilon \vee q)\)-fuzzy ideal.
Remark 3.18 Consider a semigroup $S = \{a, b, c, d\}$ in Example 3.5 and let $\lambda$ be an $(\varepsilon, \varepsilon \lor q_0^\delta)$-fuzzy left ideal of $S$. Then

$$\lambda(a) = \begin{cases} \varepsilon & \text{if } x = a, \\ \frac{3\delta}{5} & \text{if } x = b, \\ \frac{7\delta}{10} & \text{if } x = c, \\ 0 & \text{if } x = d, \end{cases}$$

and

$$\nu(a) = \begin{cases} \frac{10\delta}{7} & \text{if } x = a, \\ \frac{\delta}{2} & \text{if } x = b, \\ \frac{3\delta}{5} & \text{if } x = c, \\ \frac{\delta}{5} & \text{if } x = d, \end{cases}$$

respectively. Then $\lambda$ and $\nu$ are $(\varepsilon, \varepsilon \lor q_0^\delta)$-fuzzy ideals of $S$ by Remark 3.18. It follows from Theorems 3.13 and 3.15 that $\lambda \circ \nu$ and $\lambda \land \nu$ are $(\varepsilon, \varepsilon \lor q_0^\delta)$-fuzzy ideals of $S$. Moreover, we know that $\lambda \circ \nu \neq \lambda \land \nu$ since

$$(\lambda \circ \nu)(b) = \bigvee_{b=xy} \min\{\lambda(x), \nu(y)\} = \frac{3\delta}{5} > \frac{\delta}{2} = (\lambda \land \nu)(b).$$

Lemma 3.20 A fuzzy set $\lambda$ in $S$ is an $(\varepsilon, \varepsilon \lor q_0^\delta)$-fuzzy generalized bi-ideal of $S$ if and only if the following assertion is valid.

$$\lambda(xyz) \geq \lambda(x) \land \lambda(z) \lor q_0^\delta$$
Proof. Let $\lambda$ be an $(\varepsilon, \in \lor q_0^\delta)$-fuzzy generalized bi-ideal of $S$ and assume that the condition (8) is false, that is, there exist $a, b, c \in S$ and $t \in (0, \delta]$ such that

$$\lambda(abc) < t \leq \min\{\lambda(a), \lambda(c), \frac{\delta}{2}\}.$$  

Then $a_t \in \lambda$, $c_t \in \lambda$, $(abc)_t \in \lambda$ and $\lambda(abc) + t < 2t \leq \delta$. This is a contradiction, and so the condition (8) is valid.

Conversely, assume that $\lambda(xyz) \geq \min\{\lambda(x), \lambda(z), \frac{\delta}{2}\}$ for all $x, y, z \in S$. Let $t$ and $r$ be elements of $(0, \delta]$ such that $x_t \in \lambda$ and $z_r \in \lambda$. Then $\lambda(x) \geq t$ and $\lambda(z) \geq r$, which imply from (8) that

$$\lambda(xyz) \geq \min\{\lambda(x), \lambda(z), \frac{\delta}{2}\} \geq \min\{t, r, \frac{\delta}{2}\}$$

$$= \begin{cases} 
\min\{t, r\} & \text{if } t \leq \frac{\delta}{2} \text{ or } r \leq \frac{\delta}{2}, \\
\frac{\delta}{2} & \text{if } t > \frac{\delta}{2} \text{ and } r > \frac{\delta}{2}.
\end{cases}$$

Hence $(xyz)_{\min\{t, r\}} \in \lambda$ or $\lambda(xyz) + \min\{t, r\} > \frac{\delta}{2} + \frac{\delta}{2} = \delta$, that is, $(xyz)_{\min\{t, r\}} \in \lor q_0^\delta \lambda$. Therefore $\lambda$ is an $(\varepsilon, \in \lor q_0^\delta)$-fuzzy generalized bi-ideal of $S$.

Corollary 3.21 ([11]) A fuzzy set $\lambda$ in $S$ is an $(\varepsilon, \in \lor q)$-fuzzy generalized bi-ideal of $S$ if and only if $\lambda(xyz) \geq \min\{\lambda(x), \lambda(z), 0.5\}$ for all $x, y, z \in S$.

Combining Lemmas 3.2 and 3.20, we have the following theorem.

Theorem 3.22 A fuzzy set $\lambda$ in $S$ is an $(\varepsilon, \in \lor q_0^\delta)$-fuzzy bi-ideal of $S$ if and only if it satisfies (3) and (8).

In general, any $(\varepsilon, \in \lor q)$-fuzzy generalized bi-ideal of $S$ may not be an $(\varepsilon, \in \lor q)$-fuzzy bi-ideal of $S$. In fact, the $(\varepsilon, \in \lor q)$-fuzzy generalized bi-ideal $\lambda$ of $S$ in Example 3.5 is not an $(\varepsilon, \in \lor q)$-fuzzy bi-ideal of $S$ since $\lambda(cc) = \lambda(b) = \frac{\delta}{10} \neq \frac{\delta}{5} = \min\{\lambda(c), \lambda(c), \frac{\delta}{2}\}$.

Now, we consider conditions for an $(\varepsilon, \in \lor q)$-fuzzy generalized bi-ideal to be an $(\varepsilon, \in \lor q)$-fuzzy bi-ideal.

Theorem 3.23 In a regular semigroup $S$, every $(\varepsilon, \in \lor q_0^\delta)$-fuzzy generalized bi-ideal is an $(\varepsilon, \in \lor q_0^\delta)$-fuzzy bi-ideal.

Proof. Let $\lambda$ be an $(\varepsilon, \in \lor q_0^\delta)$-fuzzy generalized bi-ideal of a regular semigroup $S$. Let $a$ and $b$ be any elements of $S$. Then $b = bxb$ for some $x \in S$. It follows from (8) that $\lambda(ab) = \lambda(a(bxb)) = \lambda(a(bx)b) \geq \min\{\lambda(a), \lambda(b), \frac{\delta}{2}\}$ and so that $\lambda$ is an $(\varepsilon, \in \lor q_0^\delta)$-fuzzy bi-ideal of $S$ by Theorem 3.22.

Corollary 3.24 ([11]) Every $(\varepsilon, \in \lor q)$-fuzzy generalized bi-ideal of a regular semigroup $S$ is an $(\varepsilon, \in \lor q)$-fuzzy bi-ideal of $S$. 

A novel generalization of fuzzy ideals 2545
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