Modeling of the Impact Response of a Beam in a Viscoelastic Medium

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Abstract

In the present paper, we consider the problem on a transverse impact of a viscoelastic sphere upon an elastic Bernoulli-Euler beam in a viscoelastic medium, the viscoelastic features of which are described by certain viscoelastic operators. The Young's modulus of the impactor is the time-dependent operator, which is defined via the standard linear solid fractional derivative model, while the bulk modulus is considered to be constant. Within the contact domain the contact force is defined by the modified Hertzian contact law with the time-dependent rigidity function. For decoding the viscoelastic operators involving in the problem under consideration, the algebra of Rabotnov's fractional operators is employed. The integral equation is obtained for the contact force.

Keywords: Impact response, Bernoulli-Euler beam, Rabotnov's fractional operators, Green function, standard linear solid fractional derivative model

1 Introduction

Dynamic response of fractionally damped beams and other engineering structures has been considered by several researchers, see for example, [1]-[3], but papers dealing with the impact response of viscoelastic engineering structures, damping features of which are described via fractional calculus models, are rather rare [4]-[6].
The main goal of the present paper is to formulate the problem on transverse impact of a viscoelastic spherical impactor upon an elastic beam in a viscoelastic medium for the case, when the viscoelastic features of the impactor are described by the fractional derivative standard linear solid model, while the damping features of the surrounding medium are modelled by the fractional derivative Kelvin-Voigt model [7] with the different fractional parameter.

2 Problem formulation

Let us consider the problem on a transverse impact of a viscoelastic sphere upon a viscoelastic Bernoulli-Euler beam, when the viscoelastic features of the target are described by a fractional derivative Kelvin-Voigt model. In this case, the equations of motion of a spherical impactor of radius $R$ and the viscoelastic beam of length $L$ have, respectively, the form

$$m\ddot{w}_2 = -P(t),$$

$$E_l \frac{\partial^4 w_1}{\partial x^4} + qA\ddot{w}_1 = P(t)\delta\left(x - \frac{L}{2}\right)$$

where $m$ is the mass of the sphere, $w_2$ is the displacement of the sphere, $P(t)$ is the contact force, $w_1(x,t)$ is the displacement of the beam at the contact point, $I$ is the moment of inertia of the beam's cross section, $A$ is the beam's cross sectional area, $\rho$ is its density, $\delta(x - \frac{L}{2})$ is the Dirac delta-function, $x$ is the longitudinal coordinate, an overdot denotes partial time-derivative, $E$ is the viscoelastic operator

$$E = E_1\left(1 + \tau_{\sigma_1}^{\gamma_1}D^{\gamma_1}\right),$$

$E_1$ is the relaxed elastic modulus, $\tau_{\sigma_1}$ is the retardation time, $\gamma_1$ ($0 < \gamma_1 \leq 1$) is the fractional parameter, i.e. the order of the Riemann-Liouville fractional derivative

$$D^{\gamma_1}x(t) = \frac{d}{dt}\int_0^t \frac{(t-t')^{-\gamma_1}}{\Gamma(1-\gamma_1)} x(t')dt',$$

and $\Gamma(1-\gamma_1)$ is the Gamma-function.

Equations (1) and (2) are subjected to the following initial conditions:

$$w_1(x,0) = 0, \quad \dot{w}_1(x,0) = 0, \quad w_2(0) = 0, \quad \dot{w}_2(0) = V_0,$$

where $V_0$ is the initial velocity of the impactor at the moment of impact.
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Integrating twice Eq. (1) yields

\[ w_2(t) = -\frac{1}{m} \int_0^t P(t') (t - t') dt' + V_0 t. \]  

(5)

Expanding the displacement \( w_i(t) \) for a simply-supported Bernoulli-Euler beam in terms of eigenfunctions \( W_n(x) \), we have

\[ w_i(x, t) = \sum_{n=1}^{\infty} W_n(x) T_n(t), \]  

(6)

where \( T_n(t) \) are generalized displacements, and

\[ W_n(x) = \sin \left( \frac{n\pi}{L} x \right). \]  

(7)

Substituting (6) in Eq. (2) with due account for (3), and considering the orthogonality condition for the eigenfunctions (7) on the segment from 0 to \( L \), we are led to the infinite set of uncoupled equations

\[ \ddot{T}_n(t) + \Omega_n^2 \left( 1 + \tau_{\eta}^{\gamma} D^{\gamma} \right) T_n(t) = F_n P(t), \quad (n = 1, 2, \ldots) \]  

(8)

each of which describes force driven vibrations of the viscoelastic oscillator, where

\[ \Omega_n^2 = \frac{EI}{\rho A} \left( \frac{n\pi}{L} \right)^4, \quad F_n = \frac{2}{\rho AL} \sin \frac{n\pi}{L} \]  

Now let us show that Eq. (8) could be obtained as well, if we consider the problem on vibrations of an elastic beam in a viscoelastic medium. Really, the equation describing vibrations of the elastic beam in the viscoelastic medium, damping features of which are modelled by a fractional derivative, has the form

\[ \frac{EI}{\rho A} \frac{\partial^4 w_i}{\partial x^4} + \frac{\mu}{\rho A} D^{\gamma} w_i + \ddot{w}_i = \frac{1}{\rho A} P(t) \delta \left( x - \frac{L}{2} \right), \]  

(9)

where \( \mu \) is viscous coefficient.

Substituting (6) in Eq. (9), and considering the orthogonality condition for the eigenfunctions (7) on the segment from 0 to \( L \), we are led to the infinite set of uncoupled equations

\[ \ddot{T}_n(t) + \frac{\mu_n}{\rho A} D^{\gamma} T_n(t) + \Omega_n^2 T_n(t) = F_n P(t), \quad (n = 1, 2, \ldots) \]  

(10)

where \( \mu_n \) is the viscous coefficient of the \( n \)th mode.

Considering the Rayleigh hypothesis about the proportionality between the elastic and viscous matrices, i.e.

\[ \frac{\mu_n}{\rho A} = \Omega_n^2 \tau_{\eta}^{\gamma}, \]  

where \( \tau_{\eta}^{\gamma} \) is the coefficient of proportionality, Eq. (10) is reduced to Eq. (8).

Thus, our assertion has been proven.
3 Green function for the fractional derivative Kelvin-Voigt model

In order to find the solution of Eq. (9), it is necessary to find the Green function $G_n(t)$ for each oscillator from (8)

$$G_n(t) = A_n(t) + A_n e^{-\alpha_n t} \sin(\omega_n t - \varphi_n),$$  \hspace{1cm} (11)

where the index $n$ indicates the ordinal number of the oscillator, and all values entering in (11) have the same structure and the same physical meaning as the corresponding values discussed in [1], i.e. $A_n$ is the amplitude, $\alpha_n$ is the damping coefficient, and $\omega_n$ and $\varphi_n$ are the frequency and phase, respectively.

Reference to Eq. (11) shows that the Green function possesses two terms, one of which, $A_n(t)$, describes the drift of the equilibrium position and is represented by the integral involving the distribution function of dynamic and rheological parameters, while the other term is the product of two time-dependent functions, exponent and sine, and it describes damped vibrations around the drifting equilibrium position.

Now let us write Eq. (8) in terms of the Green function $G_n(t)$

$$\ddot{G}_n(t) + \Omega^2_n \tau^\gamma_n D^\gamma G_n(t) + \Omega^2_n G_n(t) = F_n \delta(t) \quad (n = 1, 2, \ldots).$$  \hspace{1cm} (12)

Applying the Laplace transform to Eq. (12) yields

$$\bar{G}_n = \frac{F_n}{p^2 + \kappa_n p^\gamma_n + \Omega_n^2},$$  \hspace{1cm} (13)

where an overbar denotes the Laplace transform of the corresponding function, $p$ is the transform parameter, and $\kappa_n = \Omega_n^2 \tau^\gamma_n$.

If we omit the number $n$ in (13), then it will coincide with formula (2.2.1) in Sect. 2.2 [7] devoted to the vibrations of the fractional derivative Kelvin-Voigt oscillator. All further formulas of this Section, (2.2.2)-(2.2.6), refer to the analysis of the roots of the characteristic equation

$$p^2 + \kappa_n p^\gamma_n + \Omega_n^2 = 0,$$  \hspace{1cm} (14)

which at each $n$ possesses two complex conjugate roots $(p_n, \kappa_n) = r_n e^{i\omega_n} = -\alpha_n \pm i\omega_n$ (see the root locus at $n = 1$ in Fig. 19 of [7]), and the inversion of the expression (13) on the first sheet of the Riemannian surface. If we insert the index $n$ in these formulas, then we obtain the desired relationship (9), where the function $A_n(t)$ describes the drift of the equilibrium position

$$A_n(t) = \int_0^\infty \tau^{-1} B_n(\tau, \kappa_n) e^{-\gamma/\tau} d\tau,$$  \hspace{1cm} (15)
the function \( B_n(\tau, \kappa_n) \)

\[
B_n(\tau, \kappa_n) = \frac{\sin \pi \gamma_1}{\pi} \frac{F_n \pi [\theta_n(\tau)]^{-1}}{[\theta_n(\tau)]^{-1} \kappa_n^{-1} \tau^{\gamma_1-2} + \theta_n(\tau) \kappa_n \tau^{2-\gamma_1} + 2 \cos \pi \gamma_1}
\]

gives us the distribution of the creep (retardation) parameters of the dynamic system,

\[
\theta_n(\tau) = \tau^2 \Omega_n^2 + 1,
\]

and the amplitude \( A_n \) and phase \( \phi_n \) of vibrations are defined, respectively, as

\[
A_n = 2 \int_{\gamma_1} 4 r_n^2 + \gamma_1^2 \kappa_n^2 r_n^{2(\gamma_1-1)} + 4 \gamma_1 \kappa_n \gamma_1 \cos(2 - \gamma_1) \psi_n^2 \]

\[
\tan \phi_n = \frac{2 r_n \cos \psi_n + \gamma_1 \kappa_n \gamma_1 \cos(1 - \gamma_1) \psi_n}{2 r_n \sin \psi_n - \gamma_1 \kappa_n \gamma_1 \sin(1 - \gamma_1) \psi_n}.
\]

4 Determination of the contact force

Knowing the Green functions, the solution of Eq. (2) takes the form

\[
w_i(x, t) = \sum_{n=1}^{\infty} \sin \left( \frac{n \pi x}{L} \right) \int_0^t G_n(t - t') P(t') dt'.
\]

Let us introduce the value characterizing the relative approach of the sphere and beam, i.e., penetration of the elastic beam by the viscoelastic sphere, is

\[
y(t) = w_2(t) - w_1 \left( \frac{L}{2}, t \right),
\]

which is connected with the contact force by the generalized Hertzian law

\[
P(t) = \tilde{k} y^{3/2},
\]

where

\[
\tilde{k} = \frac{4}{3} \sqrt{RE^*}
\]

is the operator involving the geometry and viscoelastic features of the impactor and elastic features of the target, which are described due to the Volterra correspondence principle by the operator \( E^* \)

\[
\frac{1}{E^*} = J^* = \frac{1 - \nu_1^2}{E_1} + \frac{1 - \tilde{\nu}_2^2}{\tilde{E}_2},
\]

where \( \nu_1 \) and \( E_1 \) are constant Poisson's coefficient and Young's modulus, respectively, for the elastic beam (target), while \( \tilde{\nu}_2 \) and \( \tilde{E}_2 \) are the operators for the viscoelastic sphere (impactor).

Further in order to obtain the integro-differential equation for the values \( y(t) \) and \( P(t) \), it is necessary to assign the form of the operator \( \tilde{E}_2 \).
Assume that viscoelastic features of the impactor's material are described by the fractional derivative standard linear solid model, i.e. the operator \( \dot{E}_2 \) has the form

\[
\sigma + \tau^{\gamma_2}_{\varepsilon_2} D^{\gamma_2} \sigma = E_0 \left( \varepsilon + \tau^{\gamma_2}_{\sigma_2} D^{\gamma_2} \varepsilon \right),
\]

(21)

where \( \gamma_2 \) is impactor's fractional parameter, \( \tau^{\varepsilon_2}_{\varepsilon_2} \) and \( \tau^{\sigma_2}_{\sigma_2} \) are its relaxation and retardation times, respectively, and \( E_0 \) is the relaxed magnitude of the impactor's material elastic modulus.

Following Rabotnov [8] and Rossikhin and Shitikova [6], assume that the bulk modulus of the impactor's material is a constant value, i.e.,

\[
\frac{\dot{E}_2}{1 - 2 \tilde{v}_2} = \frac{E_\infty}{1 - 2 \nu_\infty}
\]

(22)

where \( E_\infty \) and \( \nu_\infty \) are the nonrelaxed magnitude of impactor's material elastic modulus and Poisson's ratio, respectively.

From (22) it could be found [6] that Poisson's time-dependent operator could be written in the form

\[
\tilde{\nu}_2 = \nu_\infty + \frac{1}{2} (1 - 2 \nu_\infty) \nu_\varepsilon \mathcal{Y}_\varepsilon \left( \tau^{\gamma_2}_{\varepsilon_2} \right),
\]

(23)

where \( \mathcal{Y}_\gamma \left( \tau^{\gamma_2}_{\varepsilon_2} \right) \) is the dimensionless Rabotnov's fractional operator [9]

\[
\mathcal{Y}_\gamma \left( \tau^{\gamma_2}_{\varepsilon_2} \right) = \frac{1}{1 + \tau^{\gamma_2}_{\varepsilon_2} D^{\gamma_2}} \quad (i = \varepsilon, \sigma)
\]

(24)

and

\[
\dot{E}_2 = E_\infty \left[ 1 - \nu_\varepsilon \mathcal{Y}_\varepsilon \left( \tau^{\gamma_2}_{\varepsilon_2} \right) \right]
\]

(25)

with

\[
\nu_\sigma = \frac{J_0 - J_\infty}{J_\infty}, \quad \nu_\varepsilon = \frac{J_0 - J_\infty}{J_\infty}, \quad \nu_\varepsilon = \frac{J_0 - J_\infty}{J_\infty}, \quad \nu_\sigma = \frac{J_0 - J_\infty}{J_\infty} = \frac{E_\infty - E_\infty}{J_\infty} = \frac{\tau^{\gamma_2}_{\varepsilon_2}}{\tau^{\gamma_2}_{\varepsilon_2}},
\]

(26)

and \( J_0 \) and \( J_\infty \) are relaxed and nonrelaxed impactor's compliances, respectively.

Using the algebra of dimensionless Rabotnov's fractional operators recently developed in [6,9], it is possible to decode the operator \( (1 - \tilde{v}_2^2) \dot{E}_2^{-1} \), resulting in

\[
\frac{1 - \tilde{v}_2^2}{\dot{E}_2} = \frac{1 - \nu_\varepsilon^2}{E_\infty} \left[ 1 + \frac{(1 - 2 \nu_\infty)^2 \nu_\varepsilon}{4(1 - \tilde{v}_2^2)} \mathcal{Y}_\varepsilon \left( \tau^{\gamma_2}_{\varepsilon_2} \right) + \frac{3 \nu_\sigma}{4(1 - \tilde{v}_2^2)} \mathcal{Y}_\sigma \left( \tau^{\gamma_2}_{\varepsilon_2} \right) \right]
\]

(27)
Substituting operator (27) in (20) yields

\[
\frac{1}{E^*} = \frac{1 - v_1^2}{E_1} + \frac{1 - v_2^2}{E_\infty} + \frac{(1 - 2v_\infty)^2v_\infty}{4E_\infty} \gamma_{\tau_2}^\ast \left( \tau_{\tau_2}^\ast \right) + \frac{3\nu_2}{4E_\infty} \gamma_{\tau_2}^\ast \left( \nu_{\tau_2}^\ast \right).
\] (28)

Now substituting (28) in the Hertzian contact law (18) with due account for Eqs. (10) and (17), we are led to the integral equation for defining the contact force

\[
\left( \frac{3}{4\sqrt{R}} \right)^{23} \left[ \left( \frac{1 - v_1^2}{E_1} + \frac{1 - v_2^2}{E_\infty} \right)P(t) + \frac{(1 - 2v_\infty)^2v_\infty}{4E_\infty} \int_0^t \gamma_{\tau_2} \left( -\frac{t - t'}{\tau_{\tau_2}} \right)P(t')dt' \right] + \frac{3\nu_2}{4E_\infty} \int_0^t \gamma_{\tau_2} \left( -\frac{t - t'}{\tau_{\tau_2}} \right)P(t')dt' = -\frac{1}{m} \int_0^t P(t')(t - t')dt' + V_0t
\] (29)

\[-\sum_{n=1}^\infty \sin \left( \frac{n\pi}{2} \right) \int_0^t G_n(t - t')P(t')dt',
\]

where

\[
\gamma_{\tau_2} \left( -\frac{t}{\tau_{\tau_2}} \right) = \frac{t_{\tau_2}^{\gamma_2-1}}{\tau_{\tau_2}^{\gamma_2-1}} \sum_{n=0}^\infty (-1)^n (t/\tau_{\tau_2})^{\gamma_2n} \Gamma[\gamma_2(n + 1)] (i = \varepsilon, \sigma).
\] (30)

5 Defining the local indentation. Approximate solution

In order to find the equation in terms of \( y(t) \), it is necessary to utilize relationship (17) with due account for (5), (16), (18) and (19). Since formula (19) involves the operator \( \tilde{k} \), then for its construction it is needed to inverse the operator \( \tilde{k}^{-1} = \frac{3}{4\sqrt{R}}E^{-1} \), where operator \( E^{-1} \) is defined in (28). Thus the operator \( \tilde{k} \) could be represented in the form

\[
\tilde{k} = l^{-1} \left[ 1 - e_1 \gamma_{\gamma_2} \left( \tau_{\gamma_2}^2 \right) - e_2 \gamma_{\gamma_2} \left( \tau_{\gamma_2}^2 \right) \right],
\] (31)

where constants \( e_1 \), \( e_2 \), \( t_1 \), and \( t_2 \) have been found in [8], and

\[l = \frac{1 - v_1^2}{E_1} + \frac{1 - v_2^2}{E_\infty} .\]

Now considering (31) a nonlinear integral equation for determining the value \( y(t) \) takes the form

\[
y(t) = -\frac{4\sqrt{R}}{3m} \int_0^t \left[ y^{3/2}(t') - \sum_{j=1}^2 e_j \int_0^t \gamma_{\gamma_2} \left( \tau_{\gamma_2}^2 \right) \left( -\frac{t' - t''}{t_j} \right) y^{3/2}(t'')dt'' \right] (t - t')dt' + V_0t
\]

\[-\frac{4\sqrt{R}}{3l} \sum_{n=1}^\infty \sin \left( \frac{n\pi}{2} \right) \int_0^t G_n(t - t') \left[ y^{3/2}(t') - \sum_{j=1}^2 e_j \int_0^t \gamma_{\gamma_2} \left( \tau_{\gamma_2}^2 \right) \left( -\frac{t' - t''}{t_j} \right) y^{3/2}(t'')dt'' \right] dt'.
\] (32)
Since the impact process is of short duration, then
\[ \gamma^2 \left( \frac{t}{t_j} \right) \approx \frac{t^{\gamma^2 - 1}}{t_j^{\gamma^2} \Gamma(\gamma^2)}, \quad (33) \]
while the Green function \( G_n(t) \), which vanishes to zero at \( t = 0 \) according to the limiting theorem
\[ \lim_{p \to 0} G_n(p) p = G(0) = 0, \quad (34) \]
is represented in the form
\[ G_n(t) \approx t A_n \omega_n \cos \varphi_n, \quad (35) \]
then considering (33)-(35) Eq. (32) is reduced to
\[ y(t) = V_0 t - \frac{4\sqrt{R}}{3l} \left( \frac{1}{m} + \sum_{n=1}^{\infty} A_n \omega_n \cos \varphi_n \sin \frac{n\pi}{2} \right) \int_0^t y^{3/2}(t') \]
\[ - \sum_{j=1}^{2} \frac{e_j}{t_j^{\gamma^2} \Gamma(\gamma^2)} \int_0^t (t' - t'')^{\gamma^2 - 1} y^{3/2}(t'') dt'' \bigg] (t - t') dt'. \quad (36) \]

As a first approximation for the function \( y(t) \), the expression
\[ y = V_0 t \quad (37) \]
could be utilized. Considering (37) and the relationship
\[ \left( 1 - \frac{t''}{t'} \right)^{\gamma^2} \approx 1 - \gamma^2 \frac{t''}{t'}, \]
we could calculate the integral
\[ \int_0^t (t' - t'')^{\gamma^2 - 1} y^{3/2}(t'') dt'' = \frac{V_0^{3/2}}{\gamma^2} \int_0^t (t'')^{\gamma^2} d(t' - t'')^{\gamma^2} \]
\[ = \frac{V_0^{3/2}}{\gamma^2} \int_0^t (t'')^{\gamma^2} \left( 1 - \frac{t''}{t'} \right)^{\gamma^2} \left( t'' \right)^{1/2} dt'' = \frac{3V_0^{3/2}}{2 \gamma^2} \int_0^t (t'')^{\gamma^2} \left( 1 - \frac{t''}{t'} \right)^{1/2} \left( t'' \right)^{1/2} dt'' \quad \left(38\right) \]
\[ = \frac{3V_0^{3/2}}{\gamma^2} \left( 1 - \frac{1}{5\gamma^2} \right) (t')^{3/2+\gamma^2}. \]

Now substituting (37) and (38) in the right-hand side of (36) yields
\[ y(t) = V_0 t - \frac{4}{35} \Delta_1 \gamma_0 V_0^{3/2} 7^{3/2} + 3 \Delta_1 \gamma_2 V_0^{3/2} \frac{1/3 - 1/5 \gamma_2}{\gamma_2 (5/2 + \gamma_2)(7/2 + \gamma_2)} t^{7/2 + \gamma_2}. \quad (39) \]
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where

\[ \Delta_\gamma = \frac{4\sqrt{R}}{3l} \left( \frac{1}{m} + \sum_{n=1}^{\infty} A_n \omega_n \cos \phi_n \sin \frac{n\pi}{2} \right), \quad \delta_{\gamma_2} = \frac{1}{\Gamma(\gamma_2)} \sum_{j=1}^{\infty} e_j \gamma_j \]

Since at \( \gamma_2 \to 0 \) the value \( \sum_{j=1}^{\infty} e_j = 0 \) [8], then relationship (39) is reduced to

\[ y(t) = V_0 t - \frac{4}{35} \Delta_\gamma V_0^{3/2} t^{7/2}. \quad (40) \]

Formula (40) is valid for the case of impact of an elastic sphere upon an elastic beam. From (40) it is possible to find the contact duration by vanishing \( y(t) \) to zero. As a result we obtain

\[ t^{(0)}_{\text{cont}} = \left( \frac{35}{4} \Delta_\gamma \sqrt{V_0} \right)^{2/5}. \quad (41) \]

Equating \( \frac{dy}{dt} \) to zero, we obtain the magnitude of the value \( t = t^{(0)}_{\max} \) at which \( y(t) \) attains its maximal value \( y^{(0)}_{\max} \)

\[ t^{(0)}_{\max} = \left( \frac{5}{2} \Delta_\gamma \sqrt{V_0} \right)^{2/5}, \quad (42) \]

resulting in

\[ y^{(0)}_{\max} = \frac{5}{7} V^{(0)}_{\max}. \quad (43) \]

Now we consider the case \( \gamma_2 \neq 0 \). Assuming that in this case all characteristic values differ a little from the corresponding values at \( \gamma_2 = 0 \) yields

\[ t^{(\gamma_2)}_{\text{cont}} = t^{(0)}_{\text{cont}} \left[ 1 + \frac{15}{2} \delta_{\gamma_2} \left( t^{(0)}_{\text{cont}} \right)^{\gamma_2} \frac{1/3 - 1/5 \gamma_2}{\gamma_2 (5/2 + \gamma_2)(7/2 + \gamma_2)} \right], \quad (44) \]

\[ t^{(\gamma_2)}_{\max} = t^{(0)}_{\max} \left[ 1 + 3 \delta_{\gamma_2} \left( t^{(0)}_{\max} \right)^{\gamma_2} \frac{1/3 - 1/5 \gamma_2}{\gamma_2 (5/2 + \gamma_2)} \right], \quad (45) \]

\[ y^{(\gamma_2)}_{\max} = y^{(0)}_{\max} + \frac{15}{2} \delta_{\gamma_2} V^{(0)}_{\max} \frac{1/3 - 1/5 \gamma_2}{\gamma_2 (5/2 + \gamma_2)(7/2 + \gamma_2)} \left( t^{(0)}_{\max} \right)^{1+\gamma_2}. \quad (46) \]

At the limiting case \( \gamma_2 = 1 \), i.e. in the case of conventional viscosity, formulas (44)-(46) are reduced to
\[ t_{\text{cont}}^{(1)} = t_{\text{cont}}^{(0)} \left( 1 + \frac{4}{63} \delta t_{\text{cont}}^{(0)} \right), \quad (47) \]
\[ t_{\text{max}}^{(1)} = t_{\text{max}}^{(0)} \left( 1 + \frac{4}{35} \delta t_{\text{max}}^{(0)} \right), \quad (48) \]
\[ y_{\text{max}}^{(1)} = y_{\text{max}}^{(0)} + \frac{4}{63} V_0 \left( t_{\text{max}}^{(0)} \right)^2. \quad (49) \]

Reference to the above formulas shows that with the increase in the fractional parameter \( \gamma_2 \) from 0 to 1 the viscosity of the impactor increases from 0 to its maximal magnitude, resulting in the increase of such characteristic values as the time of contact, the time needed the impactor's penetration to achieve its maximal magnitudes, and the maximal value of the local bearing of impactor and target's materials itself. The enumerated values increase from the magnitudes \( t_{\text{cont}}^{(0)}, t_{\text{max}}^{(0)}, y_{\text{max}}^{(0)} \) to \( t_{\text{cont}}^{(1)}, t_{\text{max}}^{(1)}, y_{\text{max}}^{(1)} \), respectively.

6 Conclusion

In the present paper, the problem on transverse impact of a viscoelastic spherical impactor upon an elastic beam in a viscoelastic medium has been formulated for the case, when the viscoelastic features of the impactor are described by the fractional derivative standard linear solid model, while the damping features of the surrounding medium are modelled by the fractional derivative Kelvin-Voight model with the different fractional parameter. The Green function for the target was constructed, what allows us to obtain the integral equation for the contact force using the algebra of the Rabotnov's fractional operators.

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