Common Fixed Point Theorems for Non-compatible Properties Using Implicit Functions on IFMS

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Abstract

In this paper, we prove a common fixed point theorems for non-compatible and discontinuous maps in an IFMS with implicit functions. We improve, extend and generalize the results and methods of Sharma [8] and Park [6].

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1. Introduction

In 1965, Zadeh [10] introduced the concept of fuzzy sets as a new way to represent vagueness in our life. George and Veeramani [1], Kaleva and Seikkala[4], Kramosil and Michalek [3] have introduced the concept of fuzzy metric spaces in different methods. Many authors have studied the fixed point theory in these fuzzy metric spaces( [1],[7] ). Junck [2] established common fixed point theorem for commuting maps.

Park et al. [5] defined the concept of IFMS, and proved the fixed point theorems in an IFMS. Also, Park [6] studied some common fixed point theorems for the weakly commuting maps on IFMS.

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In this paper, we prove a common fixed point theorems for non-compatible and discontinuous properties using implicit functions in an IFMS. We improve, extend and generalize the methods and results of Park [6] and Sharma [8].

2. Preliminaries

Let us recall (see [9]) that a continuous $t-$norm is a binary operation $*: [0, 1] \times [0, 1] \to [0, 1]$ which satisfies the following conditions: (a) $*$ is commutative and associative; (b) $*$ is continuous; (c) $a * 1 = a$ for all $a \in [0, 1]$; (d) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ $(a, b, c, d \in [0, 1])$.

Similarly, a continuous $t-$conorm is a binary operation $\diamond : [0, 1] \times [0, 1] \to [0, 1]$ which satisfies the following conditions: (a) $\diamond$ is commutative and associative; (b) $\diamond$ is continuous; (c) $a \diamond 0 = a$ for all $a \in [0, 1]$; (d) $a \diamond b \geq c \diamond d$ whenever $a \leq c$ and $b \leq d$ $(a, b, c, d \in [0, 1])$.

**Definition 2.1.** ([5]) The 5-tuple $(X, M, N, *, \diamond)$ is said to be an intuitionistic fuzzy metric space (shortly, IFMS) if $X$ is an arbitrary set, $*$ is a continuous $t-$norm, $\diamond$ is a continuous $t-$conorm and $M, N$ are fuzzy sets on $X^2 \times (0, \infty)$ satisfying the following conditions; for all $x, y, z \in X$, such that

(a) $M(x, y, t) > 0$,
(b) $M(x, y, t) = 1 \iff x = y$,
(c) $M(x, y, t) = M(y, x, t)$,
(d) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
(e) $M(x, y, \cdot) : (0, \infty) \to (0, 1]$ is continuous,
(f) $N(x, y, t) > 0$,
(g) $N(x, y, t) = 0 \iff x = y$,
(h) $N(x, y, t) = N(y, x, t)$,
(i) $N(x, y, t) \diamond N(y, z, s) \geq N(x, z, t + s)$,
(j) $N(x, y, \cdot) : [0, \infty) \to [0, 1]$ is continuous.

Note that $(M, N)$ is called an intuitionistic fuzzy metric on $X$. The functions $M(x, y, t)$ and $N(x, y, t)$ denote the degree of nearness and the degree of non-nearness between $x$ and $y$ with respect to $t$, respectively.

**Definition 2.2.** ([6]) Let $X$ be an intuitionistic fuzzy metric space.

(a) $\{x_n\}$ is called a Cauchy sequence if for each $\epsilon > 0$, $t > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$M(x_m, x_n, t) > 1 - \epsilon, \quad N(x_m, x_n, t) < \epsilon$$

for all $m, n \geq n_0$, where $\mathbb{N}$ is the set of natural numbers.

(b) $\{x_n\}$ is called a Cauchy sequence if

$$\lim_{n \to \infty} M(x_{n+p}, x_n, t) = 1, \quad \lim_{n \to \infty} N(x_{n+p}, x_n, t) = 0.$$

(c) $\{x_n\}$ is said to be convergent to a point $x \in X$ if, for all $t > 0$,

$$\lim_{n \to \infty} M(x_n, x, t) = 1, \quad \lim_{n \to \infty} N(x_n, x, t) = 0.$$

(d) $X$ is complete if and only if every Cauchy sequence converges in $X$. 
Definition 2.3. ([6]) Let $X$ be an IFMS and let $f, g$ be self maps of $X$.

(a) The maps $f$ and $g$ are said to be compatible if

\[
\lim_{n \to \infty} M(fgx_n, gfx_n, t) = 1, \quad \lim_{n \to \infty} N(fgx_n, gfx_n, t) = 0
\]

for all $t > 0$, whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = z$ for some $z \in X$.

(b) $(f, g)$ is said to be weakly commuting if, for all $x \in X$ and $t > 0$,

\[
M(fgx, gfx, t) \geq M(fx, gx, t), \quad N(fgx, gfx, t) \leq N(fx, gx, t).
\]

(c) $(f, g)$ is said to be $k$-weakly commuting if there exists some $k > 0$ such that, for all $x \in X$ and $t > 0$,

\[
M(fgx, gfx, t) \geq M(fx, gx, \frac{t}{k}), \quad N(fgx, gfx, t) \leq N(fx, gx, \frac{t}{k}).
\]

(d) $(f, g)$ is said to be type$(A_f)$ $k$-weakly commuting if there exists some $k > 0$ such that, for all $x \in X$ and $t > 0$,

\[
M(fgx, gfx, t) \geq M(gx, fx, \frac{t}{k}), \quad N(fgx, gfx, t) \leq N(gx, fx, \frac{t}{k}).
\]

(e) $(f, g)$ is said to be type$(A_g)$ $k$-weakly commuting if there exists some $k > 0$ such that, for all $x \in X$ and $t > 0$,

\[
M(gfx, ffx, t) \geq M(gx, fx, \frac{t}{k}), \quad N(gfx, ffx, t) \leq N(gx, fx, \frac{t}{k}).
\]

(f) A point $x \in X$ is called a coincidence point of $f$ and $g$ if and only if $fx = gx$.

Example 2.4. Let $X = [0, 2]$ with the metric $d$ defined by $d(x, y) = |x - y|$. For each $t \in (0, \infty)$, define for $x, y \in X$, $M(x, y, t) = \frac{t}{t + d(x, y)}$, $N(x, y, t) = \frac{d(x, y)}{t + d(x, y)}$, $M(x, y, 0) = 0$, $N(x, y, 0) = 1$, then $M, N$ are IFM on $X$ where is defined by $a \ast b = \min\{a, b\}$, $a \diamond b = \max\{a, b\}$.

Define $f, g : X \to X$ by

\[
f \left( \begin{array}{cc}
x & \text{if } x \in [0, \frac{1}{4}), \\
\frac{1}{4} & \text{if } x \geq \frac{1}{4},
\end{array} \right. \quad gx = \frac{x}{1+x} \text{ for all } x \in [0, 2].
\]

Consider the sequence $\{x_n = \frac{1}{3} + \frac{1}{n}\}_{n \geq 1}$ in $X$. Then $\lim_{n \to \infty} fx_n = \frac{1}{4}$, $\lim_{n \to \infty} gx_n = \frac{1}{4}$, but

\[
\lim_{n \to \infty} M(fgx_n, gfx_n, t) = \frac{t}{t + |\frac{1}{4} - \frac{1}{5}|} \neq 1,
\]

\[
\lim_{n \to \infty} N(fgx_n, gfx_n, t) = \frac{|\frac{1}{4} - \frac{1}{5}|}{t + |\frac{1}{4} - \frac{1}{5}|} \neq 0.
\]
Thus $f$ and $g$ are non-compatible maps. If taking $t = 1$ and $x = \frac{1}{5}$, then

$$M(fg(\frac{1}{5}), gg(\frac{1}{5}), t) = \frac{1}{1 + |\frac{1}{6} - \frac{1}{7}|} = \frac{42}{43},$$

$$N(fg(\frac{1}{5}), gg(\frac{1}{5}), t) = \frac{|\frac{1}{6} - \frac{1}{7}|}{1 + |\frac{1}{6} - \frac{1}{7}|} = \frac{1}{43}$$

and $M(fx, gx, \frac{1}{k}) = \frac{30}{30+k}$, $N(fx, gx, \frac{1}{k}) = \frac{k}{30+k}$. Hence for $k \geq \frac{5}{7}$, $f$ and $g$ are type($A_f$) $k$-weakly commuting at $x = \frac{1}{5}$.

3. Main Results

Theorem 3.1. Let $X$ be an IFMS with $t * t \geq t$, $t \circ t \leq t$ for all $t \in [0, 1]$ and let $f, g$ be self maps such that $f$ and $g$ are type($A_f$) $k$-weakly commuting maps or type($A_g$) $k$-weakly commuting maps at coincidence points and satisfying the following conditions: for all $x, y \in X$,

(1) $M(fx, fy, t) \geq \phi(M(gx, gy, t))$, $N(fx, fy, t) \leq \psi(N(gx, gy, t))$,

where $\phi, \psi : [0, 1] \rightarrow [0, 1]$ are continuous functions such that $\phi(t) > t$ and $\psi(t) < t$ for each $0 < t < 1$. If $f(X) \subseteq g(X)$, and if one of $f(X)$ or $g(X)$ is complete in $X$, then $f$ and $g$ have a unique common fixed point.

Proof. Let $x_0$ be an arbitrary point in $X$. Since $f(X) \subseteq g(X)$, we can choose $x_1 \in X$ such that $fx_0 = gx_1$. Generally, chosen $x_{n+1}$ such that $fx_n = gx_{n+1}$ for $n = 0, 1, 2, \cdots$. Then for $t > 0$,

$$\begin{align*}
M(fx_n, fx_{n+1}, t) &\geq \phi(M(gx_n, gx_{n+1}, t)) \\
&= \phi(M(fx_{n-1}, fx_n, t)) > M(fx_{n-1}, fx_n, t),
\end{align*}$$

(2) $N(fx_n, fx_{n+1}, t) \leq \psi(N(gx_n, gx_{n+1}, t))$

$$\begin{align*}
&= \psi(N(fx_{n-1}, fx_n, t)) < N(fx_{n-1}, fx_n, t).
\end{align*}$$

Hence, $\{M(fx_n, fx_{n+1}, t)\}_{n \geq 0}$ is an increasing in $[0, 1]$ and $\{N(fx_n, fx_{n+1}, t)\}_{n \geq 0}$ is decreasing in $[0, 1]$.

Now, we prove that

$$\lim_{n \to \infty} M(fx_n, fx_{n+1}, t) = 1, \quad \lim_{n \to \infty} N(fx_n, fx_{n+1}, t) = 0.$$  

If $\lim_{n \to \infty} M(fx_n, fx_{n+1}, t) < 1$ and $\lim_{n \to \infty} N(fx_n, fx_{n+1}, t) > 0$, from (2),

$$\begin{align*}
\lim_{n \to \infty} M(fx_n, fx_{n+1}, t) &\geq \lim_{n \to \infty} \phi(M(gx_n, gx_{n+1}, t)) > \lim_{n \to \infty} M(fx_{n-1}, fx_n, t), \\
\lim_{n \to \infty} N(fx_n, fx_{n+1}, t) &\leq \lim_{n \to \infty} \psi(N(gx_n, gx_{n+1}, t)) < \lim_{n \to \infty} N(fx_{n-1}, fx_n, t),
\end{align*}$$

which is a contradiction. Hence

$$\lim_{n \to \infty} M(fx_n, fx_{n+1}, t) = 1, \quad \lim_{n \to \infty} N(fx_n, fx_{n+1}, t) = 0.$$
Now, for any positive integer \( p \),

\[
M(fx_n, fx_{n+p}, t) \geq M(fx_n, fx_{n+1}, \frac{t}{k}) \cdot \cdots \cdot M(fx_{n+p-1}, fx_{n+p}, \frac{t}{k}),
\]

\[
N(fx_n, fx_{n+p}, t) \leq N(fx_n, fx_{n+1}, \frac{t}{k}) \cdot \cdots \cdot N(fx_{n+p-1}, fx_{n+p}, \frac{t}{k}).
\]

Therefore

\[
\lim_{n \to \infty} M(fx_n, fx_{n+p}, t) \geq 1 \cdot 1 \cdot \cdots \cdot 1 \geq 1,
\]

\[
\lim_{n \to \infty} N(fx_n, fx_{n+p}, t) \leq 0 \cdot 0 \cdot \cdots \cdot 0 \leq 0.
\]

Thus \( \{ fx_n \} = \{ gx_{n+1} \} \) is a Cauchy sequence. Suppose that \( g(X) \) is complete, \( \lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = z \) for some \( z \in X \). Let \( u = g^{-1}z \), then \( gu = z \).

By (1), we have

\[
M(fx_n, fu, t) \geq \phi(M(gx_n, gu, t)),
\]

\[
N(fx_n, fu, t) \leq \psi(N(gx_n, gu, t)).
\]

Therefore, as \( n \to \infty \), we have

\[
M(z, fu, t) \geq 1, \quad N(z, fu, t) \leq 0,
\]

which implies that \( fu = z \), that is, \( fu = gu = z \). If \( f(X) \) is complete, then \( z \in f(X) \subset g(X) \) and we have \( fu = gu = z \). Hence \( u \) is coincidence point of \( f \) and \( g \). Also, if \( f \) and \( g \) are type(\( A_f \)) \( k \)-weakly commuting maps at coincidence point, then we have

\[
M(fgu, ggu, t) \geq M(fu, gu, \frac{t}{k}) = 1,
\]

\[
N(fgu, ggu, t) \leq N(fu, gu, \frac{t}{k}) = 0,
\]

which implies \( fgu = ggu \). Hence \( fz = gz \).

Similarly, if \( f \) and \( g \) are type(\( A_g \)) \( k \)-weakly commuting at coincidence point, \( fz = gz \). By (1), we have

\[
M(fx_n, fz, t) \geq \phi(M(gx_n, gz, t)) > M(gx_n, gz, t),
\]

\[
N(fx_n, fz, t) \leq \psi(N(gx_n, gz, t)) < N(gx_n, gz, t).
\]

We get as \( n \to \infty \),

\[
M(z, fz, t) > M(z, fz, t), \quad N(z, fz, t) < N(z, fz, t),
\]

which is a contradiction. Hence \( fz = z = gz \). Hence \( z \) is a common fixed point of \( f \) and \( g \).
If \( w(w \neq z) \) is another common fixed point of \( f \) and \( g \). Then there exists \( t > 0 \) such that
\[
M(z, w, t) = M(fz, fw, t) \geq \phi(M(gz, gw, t)) \\
\geq \phi(M(z, w, t)) > M(z, w, t),
\]
\[
N(z, w, t) = N(fz, fw, t) \leq \psi(N(gz, gw, t)) \\
\leq \psi(N(z, w, t)) < N(z, w, t),
\]
which is a contradiction. Therefore \( z = w \). Hence \( z \) is a unique common fixed point of \( f \) and \( g \).

\[\square\]

**Theorem 3.2.** Let \( X \) be an IFMS with \( t \ast t \geq t \) and \( t \circ t \leq t \) for all \( t \in [0, 1] \). Let \( f \) and \( g \) be self maps of \( X \) such that \( f \) and \( g \) are type\((A_f)\) \( k\)-weakly commuting maps or type\((A_g)\) \( k\)-weakly commuting maps at coincidence points and satisfying the following conditions:

(a) for given \( \epsilon \in (0, 1) \), there exists \( \delta \in (0, \epsilon) \) such that
\[
\epsilon \geq M(gx, gy, t) > \epsilon - \delta \Rightarrow M(fx, fy, t) > \epsilon,
\]
\[
\epsilon - \delta \leq N(gx, gy, t) < \epsilon \Rightarrow N(fx, fy, t) < \epsilon - \delta,
\]

(b) \( fx = fy \), whenever \( gx = gy \).

If \( f(X) \subseteq g(X) \) and one of \( f(X) \) or \( g(X) \) is complete, then \( f \) and \( g \) have a unique common fixed point in \( X \).

**Proof.** We can define a sequence \( \{x_n\} \subset X \) such that \( fx_n = gx_{n+1} \) for \( n = 1, 2, \ldots \). By (a), for all \( x, y \in X \) with \( gx \neq gy \), we have
\[
M(fx, fy, t) > M(gx, gy, t), \quad N(fx, fy, t) < N(gx, gy, t).
\]

Thus
\[
M(fx_n, fx_{n+1}, t) > M(gx_n, gx_{n+1}, t) = M(fx_{n-1}, fx_n, t),
\]
\[
N(fx_n, fx_{n+1}, t) < N(gx_n, gx_{n+1}, t) = N(fx_{n-1}, fx_n, t).
\]

Therefore \( \{M(fx_n, fx_{n+1}, t)\}_{n \geq 1} \) is an increasing sequence, \( \{N(fx_n, fx_{n+1}, t)\}_{n \geq 1} \) is a decreasing sequence. Since
\[
\lim_{n \to \infty} M(fx_n, fx_{n+1}, t) \leq 1, \quad \lim_{n \to \infty} N(fx_n, fx_{n+1}, t) \geq 0,
\]
we prove that
\[
\lim_{n \to \infty} M(fx_n, fx_{n+1}, t) = 1, \quad \lim_{n \to \infty} N(fx_n, fx_{n+1}, t) = 0.
\]

If \( M(fx_n, fx_{n+1}, t) < 1 \) and \( N(fx_n, fx_{n+1}, t) > 0 \) for given small \( \delta > 0 \), there exist a positive real number \( N_0 \) such that for all \( m \geq N_0 \),
\[
\lim_{n \to \infty} M(fx_n, fx_{n+1}, t) \geq M(fx_m, fx_{m+1}, t) = M(gx_{m+1}, fx_{m+2}, t) \\
> \lim_{n \to \infty} M(fx_n, fx_{n+1}, t) - \delta,
\]
\[
\lim_{n \to \infty} N(fx_n, fx_{n+1}, t) - \delta \leq N(fx_m, fx_{m+1}, t) = N(gx_{m+1}, fx_{m+2}, t) \\
< \lim_{n \to \infty} N(fx_n, fx_{n+1}, t).
\]
We obtain
\[ M(fx_m, fx_{m+1}, t) > \lim_{n \to \infty} M(fx_n, fx_{n+1}, t), \]
\[ N(fx_m, fx_{m+1}, t) < \lim_{n \to \infty} N(fx_n, fx_{n+1}, t), \]
which is a contradiction. Therefore, we have
\[ \lim_{n \to \infty} M(fx_m, fx_{m+1}, t) = \lim_{n \to \infty} M(gx_{m+1}, gx_{m+2}, t), \]
\[ \lim_{n \to \infty} N(fx_m, fx_{m+1}, t) = \lim_{n \to \infty} N(gx_{m+1}, gx_{m+2}, t). \]
Hence by the same argument, \( \{fx_n\} = \{gx_{n+1}\} \) is Cauchy sequence. Suppose that \( g(X) \) is complete, for some \( z \in X \),
\[ \lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_{n+1} = z. \]
Let \( u = g^{-1}z \), then \( gu = z \). By (a), we have
\[ M(fx_n, fu, t) \geq \phi(M(gx_n, gu, t)), \quad N(fx_n, fu, t) \leq \psi(N(gx_n, gu, t)). \]
Then we have as \( n \to \infty \), \( M(z, fu, t) \geq 1, N(z, fu, t) \leq 0 \) which implies that \( fu = z \), hence \( fu = gu = z \). If \( f(X) \) is complete, then \( z \in f(X) \subseteq g(X) \) and we have \( fu = gu = z \) that \( u \) is coincidence point of \( f \) and \( g \). Since \( f \) and \( g \) are type(\( A_f \)) k-weakly commuting at coincidence points, we have for \( k > 0 \),
\[ M(fgu, ggu, t) \geq M(fx, gx, \frac{t}{k}) = 1, \]
\[ N(fgu, ggu, t) \leq N(fx, gx, \frac{t}{k}) = 0, \]
which implies that \( fgu = ggu \). That is, \( fz = gz \). Similarly, if \( f \) and \( g \) are type(\( A_g \)) k-weakly commuting at coincidence points, we have \( fz = gz \). By (a),
\[ M(fx_n, fz, t) > \phi(M(gx_n, gz, t)), \quad N(fx_n, fz, t) < \psi(N(gx_n, gz, t)). \]
We have as \( n \to \infty \),
\[ M(z, fz, t) > M(z, gz, t), \quad N(z, fz, t) < N(z, gz, t), \]
which is contradiction. Thus \( fz = gz = z \). Hence \( z \) is a common fixed point of \( f \) and \( g \). Also, \( z \) is a unique common fixed point of \( f \) and \( g \). This completes the proof.

**Corollary 3.3.** Let \( X \) be an IFMS with \( t \ast t \geq t \) and \( t \circ t \leq t \) for all \( t \in [0, 1] \). Let \( f \) be self map of \( X \) satisfying the following condition:
(a) for given \( \epsilon \in (0, 1) \), there exists \( \delta \in (0, \epsilon) \) such that
\[ \epsilon \geq M(x, y, t) > \epsilon - \delta \quad \Rightarrow \quad M(fx, fy, t) > \epsilon, \]
\[ \epsilon - \delta \leq N(x, y, t) < \epsilon \quad \Rightarrow \quad N(fx, fy, t) < \epsilon - \delta. \]
Then \( f \) has a unique common fixed point in \( X \).

**Proof.** Let \( g = I \) be identity map from Theorem 3.2. Then the proof is complete. \( \square \)
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