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Analytical Solutions of an Economic Model by the Homotopy Analysis Method

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Abstract

The Bouali economic model is a chaotic system of three ordinary differential equations gathering the variables of profits, reinvestments and financial flow of borrowings. Using a technique developed to find analytic solutions for strongly nonlinear problems, the step homotopy analysis method (SHAM), we present the explicit series solution of this dynamically rich model from the literature. With the homotopy solution, we investigate the dynamical effect of a meaningful bifurcation parameter, the debt rate s. Given the results presented in this article, showing that the SHAM is a powerful and effective method applicable to difficult nonlinear problems in science, this work is likely to appeal to a wide audience.

Mathematics Subject Classification: Primary 34A05; Secondary 34A34

Keywords: Analytic solutions, Nonlinear differential equations, Economic model, Homotopy analysis method, Step homotopy analysis method

1 Preliminaries. Description of the model

In order to analyse relevant aspects of a firm dynamics, S. Bouali proposed recently in [2] a system of ordinary differential equations involving three dynamical variables: *profits*, *reinvestments* and *financial flow of borrowings*.

In this article, we give a brief description of the Bouali economic model and we present the notations and basic definitions of a general analytical technique - the homotopy analysis method (HAM). We show the applicability and effectiveness of HAM to the Bouali equations when treated has an algorithm in a sequence of intervals - the step homotopy analysis method (SHAM) - for finding accuratly an analytical solution.

The Bouali firm model is given by the following autonomous system of ordinary differential equations [2]

$$\frac{dx_1}{dt} = \frac{1}{v}\left(x_2 + x_3\right), \tag{1}$$

$$\frac{dx_2}{dt} = mx_1 + n\left(1 - x_1^2\right)x_2, \tag{2}$$

$$\frac{dx_3}{dt} = -x_2x_1 + sx_2. \tag{3}$$

This system consists of three dynamical variables: (i) the variable x_1 that corresponds to the *profits*, (ii) the variable x_2 that measures the *reinvestments* and (iii) the dynamical variable x_3 that represents the financial *flow of borrowings*.

The parameters of the economic model are: v, m, n, r, s and we will consider throughout our study $v = 4.0$, $m = 0.04$, $n = 0.02$, $r = 0.068$ and take the debt rate s as a control parameter $(0 \leq s \leq 0.5)$. In a real economic context, the heuristic Bouali system can be used to analyze the asymptotic behavior of the financial dynamics of a firm, where the units of time are chosen so that time could be anything from hours to days.

2 The step homotopy analysis method analytic solutions

The analytical approach of HAM will be used in a sequence of intervals, giving rise to the step homotopy analysis method (SHAM). In the following paragraph, we outline the description of SHAM applied to the Bouali equations (1) - (3).

2.1 Explicit series solution

Let us consider the Eqs. (1)-(3) subject to the initial conditions

$$x_i(0) = IC_i, \text{ which are taken in the form } x_{i,0}(t) = IC_i \text{ for } i = 1, 2, 3,$$

as our initial approximations of $x_1(t)$, $x_2(t)$ and $x_3(t)$, respectively. In our analysis, we will consider

$$IC_1 = 4.59408, \quad IC_2 = 0.41872, \quad IC_3 = -1.63438.$$

As auxiliary linear operators, we choose

$$L\left[\phi_i(t;q)\right] = \frac{\partial \phi_i(t;q)}{\partial t} + \phi_i(t;q), \quad i = 1, 2, 3,$$

with the property $L\left[C_i e^{-t}\right] = 0$, where C_i $(i = 1, 2, 3)$ are integral constants. The equations of the Bouali model lead to the following nonlinear operators $\mathcal{N}_1$, $\mathcal{N}_2$ and $\mathcal{N}_3$:

$$\mathcal{N}_1\left[\phi_1(t;q), \phi_2(t;q), \phi_3(t;q)\right] = \frac{\partial \phi_1(t;q)}{\partial t} - \frac{1}{v}\phi_2(t;q) - \frac{1}{v}\phi_2(t;q) - \frac{1}{v}\phi_3(t;q),$$

$$\mathcal{N}_2\left[\phi_1(t;q), \phi_2(t;q), \phi_3(t;q)\right] = \frac{\partial \phi_2(t;q)}{\partial t} - m\phi_1(t;q) - n\phi_2(t;q) + n\phi_1^2(t;q)\,\phi_2(t;q),$$

$$\mathcal{N}_3\left[\phi_1(t;q), \phi_2(t;q), \phi_3(t;q)\right] = \frac{\partial \phi_3(t;q)}{\partial t} + r\phi_1(t;q) - s\phi_2(t;q).$$

Considering $q \in [0, 1]$ and h the non-zero auxiliary parameter, the $zero^{th}$-order deformation equations are

$$(1-q)\,L\left[\phi_i(t;q) - x_{i,0}(t)\right] = qh\mathcal{N}_i\left[\phi_1(t;q), \phi_2(t;q), \phi_3(t;q)\right], \tag{4}$$

with $i = 1, 2, 3$ and subject to the initial conditions

$$\phi_1(0;q) = 4.59408, \quad \phi_2(0;q) = 0.41872, \quad \phi_3(0;q) = -1.63438.$$

Obviously, for $q = 0$ and $q = 1$, the above $zero^{th}$-order equations (4) have the solutions

$$\phi_1(t;0) = x_{1,0}(t),\ \phi_2(t;0) = x_{2,0}(t),\ \phi_3(t;0) = x_{3,0}(t) \tag{5}$$

and

$$\phi_1(t;1) = x_1(t),\ \phi_2(t;1) = x_2(t),\ \phi_3(t;1) = x_3(t), \text{ respectively.} \tag{6}$$

When q increases from 0 to 1, the functions $\phi_1(t;q)$, $\phi_2(t;q)$ and $\phi_3(t;q)$ vary from $x_{1,0}(t)$, $x_{2,0}(t)$ and $x_{3,0}(t)$ to $x_1(t)$, $x_2(t)$ and $x_3(t)$.

Following the theory in [3], the m^{th}-order deformation equations are

$$L\left[x_{i,m}(t) - \chi_m x_{i,m-1}(t)\right] = h\mathcal{R}_{i,m}\left[x_{1,m-1}(t), x_{2,m-1}(t), x_{3,m-1}(t)\right], \tag{7}$$

with the following initial conditions

$$x_{1,m}(0) = 0,\ x_{2,m}(0) = 0,\ x_{3,m}(0) = 0. \tag{8}$$

Defining the vector $\overrightarrow{u}_{m-1} = \left(x_{1,m-1}(t), x_{2,m-1}(t), x_{3,m-1}(t)\right)$, we derive

$$\mathcal{R}_{1,m}\left[\overrightarrow{u}_{m-1}\right] = x_{1,m-1}(t) - \frac{1}{v}x_{2,m-1}(t) - \frac{1}{v}x_{3,m-1}(t)$$

$$\begin{aligned}\mathcal{R}_{2,m}\left[\overrightarrow{u}_{m-1}\right] &= \dot{x}_{2,m-1}(t) - m\ x_{1,m-1}(t) - n\ x_{2,m-1}(t) + \\ &+n\sum_{k=0}^{m-1}\left[\left(\sum_{j=0}^{k} x_{1,k-j}(t)\ x_{1,j}(t)\right)x_{2,m-1-k}(t)\right]\end{aligned}$$

and

$$\mathcal{R}_{3,m}\left[\overrightarrow{u}_{m-1}\right] = x_{3,m-1}(t) + r\ x_{1,m-1}(t) - s\ x_{2,m-1}(t).$$

According to the notations and definitions provided above, we solve the linear Eqs (7) at initial conditions (8), for all $m \geq 1$, and we obtain

$$x_{i,m}(t) = \chi_m\ x_{i,m-1}(t) + h\ e^{-t}\int_0^t e^{\tau}\mathcal{R}_{i,m}\left[\overrightarrow{u}_{m-1}\right]d\tau,\ (i = 1,2,3). \tag{9}$$

A M^{th}-order approximate analytic solution (which corresponds to a series solution with $M+1$ terms) is given by

$$x_{i,M}(t) = x_{i,0}(t) + \sum_{m=1}^{M} x_{i,m}(t). \tag{10}$$

Within the purpose of having an effective analytical approach of Eqs. (1) - (3) for higher values of t, we use the step homotopy analysis method, in a sequence of subintervals of time step Δt and the 6-term HAM series solutions (5^{th}-order approximations)

$$x_i(t) = x_{i,0}(t) + \sum_{m=1}^{5} x_{i,m}(t), \ (i = 1, 2, 3) \tag{11}$$

at each subinterval. Accordingly to SHAM, the initial values $x_{1,0}$, $x_{2,0}$ and $x_{3,0}$ change at each subinterval, i.e., $x_i(t^*) = IC_i^* = x_{i,0}$ $(i = 1, 2, 3)$, and the initial conditions $x_{1,m}(t^*) = x_{2,m}(t^*) = x_{3,m}(t^*) = 0$ should be satisfied for all $m \geq 1$. As a consequence, the analytical approximate solution for each dynamical variable is given by

$$x_i(t) = x_i(t^*) + \sum_{m=1}^{5} x_{i,m}(t - t^*), \ (i = 1, 2, 3) \tag{12}$$

In general, we only have information about the values of $x_1(t)$, $x_2(t)$ and $x_3(t)$ at $t = 0$, but we can obtain the values of $x_1(t)$, $x_2(t)$ and $x_3(t)$ at $t = t^*$ by assuming that the new initial conditions are given by the solutions in the previous interval. Another illustration of the use of SHAM can be seen in [1].

The homotopy terms depend on both the physical variable t and the convergence control parameter h. The artificial parameter h can be freely chosen to adjust and control the interval of convergence, and even more, to increse the convergence at a reasonable rate, fortunately at the quickest rate. With the purpose of determining the interval of convergence and the optimum value of h, corresponding to each dynamical variable, we state in what follows a very recent convergence criterion addressed in [4].

2.2 Interval of convergence and optimum value from an appropriate ratio

Let us consider $k + 1$ homotopy terms $X_0(t)$, $X_1(t)$, $X_2(t)$, ... , $X_k(t)$ of an homotopy series

$$X(t) = X_0(t) + \sum_{m=1}^{+\infty} X_m(t). \tag{13}$$

Taking a time interval Ω, the ratio

$$\beta = \frac{\displaystyle\int_{\Omega} [X_k(t)]^2 \, dt}{\displaystyle\int_{\Omega} [X_{k-1}(t)]^2 \, dt}$$

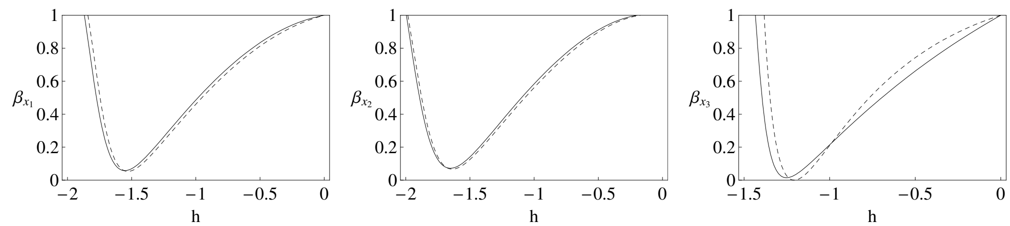

Figure 1: The curves of ratios β_{x_1}, β_{x_2} and β_{x_3} versus h. These curves correspond to a 5^{th}-order approximation of solutions $x_1(t)$, $x_2(t)$ and $x_3(t)$. The solid lines correspond to a periodic regime and the dashed lines correspond to a chaotic behavior. In each regime, the optimum value of h, h^*, gives rise to the minimum value of β. The intervals of convergence of h and the respective optimum values h^*, corresponding to the dynamical regimes are for the periodic behavior: $-1.86863 < h_{x_1} < 0$ with $h^*_{x_1} = -1.55319$, $-1.99496 < h_{x_2} < -0.19458$ with $h^*_{x_2} = -1.65225$, $-1.4337 < h_{x_3} < 0$ with $h^*_{x_3} = -1.25252$ and for the chaotic behavior: $-1.84098 < h_{x_1} < 0$ with $h^*_{x_1} = -1.52998$, $-1.98375 < h_{x_2} < -0.171187$ with $h^*_{x_2} = -1.63749$ and $-1.38241 < h_{x_3} < 0$ with $h^*_{x_3} = -1.20693$.

represents a more convinient way of evaluating the convergence control parameter h. In fact, given order of approximation, the curves of ratio β versus h indicate not only the effective region for the convergence control parameter h, but also the optimal value of h that corresponds to the minimum of β. Now, plotting β versus h, as well as by solving

$$\frac{\displaystyle\int_\Omega [X_k(t)]^2\,dt}{\displaystyle\int_\Omega [X_{k-1}(t)]^2\,dt} < 1 \quad \text{and} \quad \frac{d\beta}{dh} = 0,$$

the interval of convergence and the optimum value for the parameter h can be simultaneously achieved. As an illustration, at the order of approximation $M = 5$, the curves of ratio β versus h, corresponding to $x_1(t)$, $x_2(t)$ and $x_3(t)$, are displayed in Fig. 1. In each representation, the solid line corresponds to a periodic behavior and the dashed line corresponds to a chaotic regime.

In Fig. 2, we show the comparison between the SHAM analytical solutions, for x_1, x_2 and x_3, and the numerical results, considering the two dynamical regimes (periodic and chaotic behaviors) and using the optimum values presented in the caption of Fig. 2.

This analysis provides an illustration of how our understanding of an economic dynamical variable arising in the context of economics can be directly

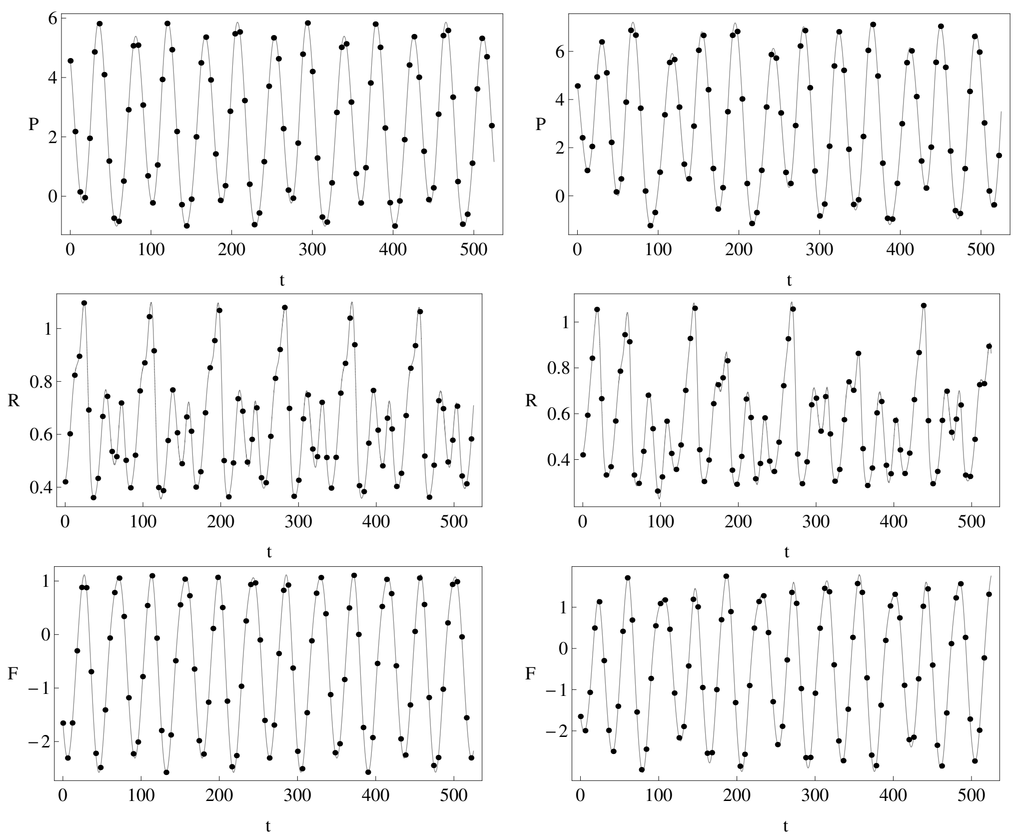

Figure 2: Comparison of the SHAM analytical solutions (12) of $P \equiv x_1$ (Profits), $R \equiv x_2$ (Reinvestments) and $F \equiv x_3$ (Flow of borrowings) - solide lines - with the respective numerical solutions - dotted lines - of the Bouali model. (Left column) Time series corresponding to the periodic regime ($s = 0.25$). (Right column) Time series corresponding to the chaotic regime ($s = 0.366$).

enhanced by the use of the respective analytical expression, for different combinations of a control parameter and time.

3 Final considerations

In this article, we applied an analytical method for nonlinear differential equations - the step homotopy analysis method (SHAM) - to solve the Bouali economic model. A new criterium of convergence presented in the literature for the control parameter h, associated to the explicit series solutions, represents a convenient way of controlling the convergence of approximate series, which is a critical qualitative difference in the analysis between HAM/SHAM and other methods. As a consequence, our results are found to be remarkably accurate, in agreement with the numerical solutions. This work is likely to inspire applications of the presented analytical procedure for solving highly nonlinear problems in different areas of knowledge.

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