Application of Generating Functions to the Theory of Success Runs

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Abstract

In the present paper, we generalize the results of the papers [2] and [4] and give a new proof of the explicit formula for the generating function of the numbers of trials before getting $k$ consecutive successes.

Mathematics Subject Classification: 12J10

Keywords: Generating function, Bernoulli scheme

1 Introduction

We consider the following well-known problem: what is the expected number of coin tosses before two consecutive heads are obtained? A similar question can be asked about a fair six-sided die: how many times do you have to roll before getting a given number $k$ of consecutive sixes? Formally, the problem can be stated as follows. We consider the Bernoulli scheme of trials in which “success” $S$ occurs with probability $p$ and “failure” $F$ occurs with probability $q = 1 - p$. Let $X$ be the random variable whose values are the numbers of trials before getting a success run of length $k$. The generating function $G_X$ of $X$ is well known (see e.g., [3, Ch. XIII, Sec.7]). Nonetheless, this problem still attracts attention of mathematicians. In the papers [2] and [4], it is shown that, in the case $p = 1/2$, the generating function $G_X$ is connected with the generating function of the so-called $k$-Fibonacci numbers. Since the generating function for these numbers is constructed in a standard way, one has a new proof of
the formula for the generating function in question. In the present paper, we generalize these results to the case of an arbitrary \( p \).

## 2 Generating function of the number of trials

We consider the sequences of trials with no success runs of length \( k \). Each such sequence can be represented as a sequence of \( F \)'s and \( S \)'s with no \( k \) successive \( S \)'s. For each \( i = 0, 1, \ldots, k - 1 \), we denote by \( H_n^i \) the event that in a sequence of \( n \) trials the outcome \( S \) occurs at the last \( i \) trials. In particular, \( H_n^0 \) means that the outcome \( F \) occurs at the last trial. Let \( u_n^{(i)} \) be the probability of \( H_n^i \).

For \( i = 0, 1, \ldots, k - 2 \), the transition

\[
\begin{array}{ccl}
\ldots F S \ldots S & \mapsto & \ldots F S \ldots S S
\end{array}
\]

happens with probability \( p \), hence \( u_n^{(i+1)} = p u_n^{(i)} \). Since the probability of \( F \) is \( q = 1 - p \), we have \( u_{n+1}^{(0)} = (1 - p) \left( u_n^{(0)} + u_n^{(1)} + \cdots + u_n^{(k-1)} \right) \). Thus, we obtain the following system of recurrence relations:

\[
\begin{align*}
& u_{n+1}^{(0)} = (1 - p) \left( u_n^{(0)} + u_n^{(1)} + \cdots + u_n^{(k-1)} \right), \\
& u_{n+1}^{(1)} = p u_n^{(0)}, \\
& u_{n+1}^{(2)} = p u_n^{(1)}, \\
& \ldots \\
& u_{n+1}^{(k-1)} = p u_n^{(k-2)}.
\end{align*}
\]

For \( n \geq k - 1 \), we put \( v_n = u_n^{(k-1)} \). Then

\[
\begin{align*}
& u_{n+1}^{(0)} = \frac{1}{p} u_n^{(1)} = \frac{1}{p^2} u_n^{(2)} = \cdots = \frac{1}{p^{k-1}} u_n^{(k-1)} = \frac{1}{p^{k-1}} v_n^{(k-1)} - 1, \\
& u_{n+1}^{(1)} = \frac{1}{p} u_n^{(2)} = \cdots = \frac{1}{p^{k-2}} u_n^{(k-1)} = \frac{1}{p^{k-2}} v_n^{(k-2)}, \\
& \ldots \\
& u_{n+1}^{(k-1)} = v_n, \\
& u_{n+1}^{(0)} = \frac{1}{p^{k-1}} v_n^{(0)},
\end{align*}
\]

which implies that

\[
\frac{v_{n+k}}{p^{k-1}} = (1 - p) \left( \frac{v_{n+k-1}}{p^{k-1}} + \frac{v_{n+k-2}}{p^{k-2}} + \cdots + v_n \right).
\]

Dividing both sides of the equation obtained by \( p^n \), we obtain

\[
\frac{v_{n+k}}{p^{n+k-1}} = (1 - p) \left( \frac{v_{n+k-1}}{p^{n+k-1}} + \frac{v_{n+k-2}}{p^{n+k-2}} + \cdots + \frac{v_n}{p^n} \right).
\]
Putting $x_n = v_n p^{-n}$, we can represent the last relation in the form

$$px_{n+k} = (1 - p)(x_{n+k-1} + x_{n+k-2} + \cdots + x_n),$$

hence

$$x_{n+k} = \frac{1-p}{p} (x_{n+k-1} + x_{n+k-2} + \cdots + x_n). \quad (1)$$

If $p = 1/2$, the relation (1) has the form

$$x_{n+k} = x_{n+k-1} + x_{n+k-2} + \cdots + x_n,$$

which, for $k = 2$, coincides with the recurrence relation for Fibonacci numbers. Relation (1) is proved for $n \geq k - 1$. For $n = 0, 1, \ldots, k - 2$, we put $x_n = 0$ and calculate the values of $x_n$ at $n = k - 1, k, \ldots, 2k - 1$. By definition, we have $x_n = v_n p^{-n} = u_n^{(k-1)} p^{-n} = P(H_n^{k-1}) p^{-n}$. The event $H_n^{k-1}$ consists of one sequence $S \ldots S$ of length $k - 1$. Therefore, its probability is equal to $p^{k-1}$, and so, $x_{k-1} = 1$. If $k \leq n \leq 2k - 1$, then the event $H_n^{k-1}$ consists of the sequences of the form $\overbrace{FS \ldots S}$, where dots at the beginning mean $F$’s or $S$’s. Consequently, \( P(H_n^{k-1}) = (1 - p)p^{k-1} \), and $x_n = (1 - p)p^{k-n-1}$ for $n = k, k + 1, \ldots, 2k - 1$.

**Lemma 2.1.** Relation (1) is valid for all integers $n \geq 0$.

*Proof.* For $n = 0$, we have $x_k = (1 - p)/p x_{k-1}$. This relation is valid since $x_k = (1 - p)/p$ and $x_{k-1} = 1$. If $1 \leq n \leq k - 1$, then $k + 1 \leq n + k \leq 2k - 1$. Therefore, $x_{n+k} = (1 - p)p^{-n-1}$. Let us calculate the right-hand side of equation (1). We have

$$x_{n+k-1} + x_{n+k-2} + \cdots + x_n = \frac{1-p}{p^n} + \frac{1-p}{p^{n-1}} + \cdots + \frac{1-p}{p} + 1 + 0 + \cdots + 0 = \frac{1-p + p - p^2 + \cdots + p^{n-1} - p^n + p^n}{p^n} = \frac{1}{p^n}.$$ 

Consequently, the right-hand side of equation (1) is equal to

$$(1 - p)p^{-n-1} = (1 - p)p^{k-n-1} = x_{n+k}.$$ 

\square

**Lemma 2.2.** The generating function of the sequence $x_n$ is given by the formula

$$\varphi(t) = \frac{t^{k-1}}{1 - \frac{1-p}{p} (t + t^2 + \cdots + t^k)} = \frac{p(t^{k-1} - 1)}{(1 - p)t^{k+1} - t + p}. \quad (2)$$
Proof. By definition, \( \varphi(t) = \sum_{n=k-1}^{\infty} x_n t^n \). Removing brackets in
\[
(x_{k-1}t^{k-1} + x_k t^k + \cdots) \left(1 - \frac{1-p}{p} \left(t + t^2 + \cdots + t^k\right)\right),
\]
we obtain by relation (1) that
\[
x_{k-1}t^{k-1} + \left(x_k - \frac{1-p}{p} x_{k-1}\right)t^k + \left(x_{k+1} - \frac{1-p}{p} (x_k + x_{k-1})\right)t^{k+1} + \cdots
\]
\[
+ \left(x_{2k-2} - \frac{1-p}{p} (x_{2k-3} + \cdots + x_{k-1})\right)t^{2k-2} \cdots
\]
\[
+ \left(x_{n+k} - \frac{1-p}{p} (x_{n+k-1} + \cdots + x_n)\right)t^{n+k} + \cdots = t^{k-1},
\]
which implies equation (2).

Lemma 2.3. We have \( G_X(t) = pt \varphi(pt) \), where \( \varphi(t) \) is the generating function of the sequence \( x_n \).

Proof. We note that the probability that \( k \) successive events \( h \) occur the first time immediately after the \( n \)th trial in Bernoulli’s scheme is the product of \( p \) and the probability of \( H_{n-1}^{k-1} \). Thus,
\[
P(X = n) = pP(H_{n-1}^{k-1}) = pu_{n-1}^{k-1} = pv_{n-1} = p^n x_{n-1}.
\]
Therefore,
\[
G_X(t) = \sum_{n=k}^{\infty} P(X = n)t^n = \sum_{n=k}^{\infty} p^n x_{n-1} t^n = pt \sum_{n=k}^{\infty} x_n(pt)^n = pt \varphi(pt).
\]

In conclusion, we derive explicit formulas for the terms of the sequence \( x_n \). We consider the polynomial
\[
g(z) = z^k - \frac{1-p}{p} (z^{k-1} + z^{k-2} + \cdots + 1).
\]
Let \( z_1, z_2, \ldots, z_k \) be the complex roots of \( g(z) \), and let
\[
a_n = \sum_{i=1}^{k} \frac{z_i^n}{g'(z_i)}.
\]
Since \( g(z) = (z - z_1)(z - z_2) \ldots (z - z_k) \), we have the relations
\[
a_0 = a_1 = \cdots = a_{k-2} = 0 \text{ and } a_{k-1} = 1.
\]
Since \( z_i \) are the roots of \( g(z) \), we obtain
\[
z_i^{n+k} = z_i^n \cdot z_i^k = \frac{1-p}{p} \left(z_i^{n+k-1} + z_i^{n+k-2} + \cdots + z_i^n\right),
\]
which implies the relation
\[ a_{n+k} = \frac{1-p}{p} (a_{n+k-1} + a_{n+k-2} + \cdots + a_n). \]

Therefore, the numbers \( a_n \) satisfy the same \( k \)th order recurrence relations as the numbers \( x_n \) do. Taking into account that the first \( k \) terms of the sequences in question coincide, we see that \( x_n = a_n \) for all \( n \geq 0 \). As a corollary, we obtain the following explicit formula for the terms of the sequence \( x_n \):
\[ x_n = \sum_{i=1}^{k} \frac{z_i^n}{g_i'(z_i)}. \]  

(3)

Now we use formula (5) from the paper [1]. Let \( w_n \) be the complete homogeneous polynomial of degree \( n \) in \( k \) variables. We put, by definition, \( w_0 = 1 \) and \( w_{-1} = w_{-2} = \cdots = w_{1-k} = 0 \). By Theorem 1 of the paper [1], we have the relation
\[ \sum_{i=1}^{k} \frac{z_i^{n+k-1}}{g_i'(z_i)} = w_n(z_1, z_2, \ldots, z_k), \]
from which we obtain by (3) that
\[ x_n = w_{n-k+1}(z_1, z_2, \ldots, z_k). \]

Acknowledgements. The first named author was partially supported by the Russian Foundation for Basic Research (grant no. 14-01-00393)

References


Received: July 6, 2016; Published: August 14, 2016