On the Optimization of Numerical Dispersion and Dissipation of Finite Difference Scheme for Linear Advection Equation

G. V. Krivovich

Faculty of Applied Mathematics and Processes of Control
Saint Petersburg State University,
7/9 Universitetskaya nab., Saint Petersburg, 199034, Russian Federation

E. S. Marnopolskaya

Faculty of Applied Mathematics and Processes of Control
Saint Petersburg State University,
7/9 Universitetskaya nab., Saint Petersburg, 199034, Russian Federation

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Abstract

Finite difference scheme for linear advection equation with dependence on scalar dimensionless parameter is constructed. Stability of the scheme is investigated by von Neumann method and stability domain in parameter space is constructed. Dispersive and dissipative properties of the scheme are optimized by the choice of scalar parameter. Low dispersion and dissipation of the scheme is demonstrated by the numerical solution of simple test Cauchy problem. Scheme constructed may be applied in computations based on splitting method for Boltzmann or lattice Boltzmann equations.

Mathematics Subject Classification: 65M12, 35E99

Keywords: lattice Boltzmann method, advection, stability, dispersion, dissipation
1 Introduction

Nowadays lattice Boltzmann method (LBM) [2], [9], [11], [12] is considered as a powerful method for the solution of different fluid dynamics problems [3], [13], [14]. The method is based on the solution of the system of kinetic equations instead the system of the equations of hydrodynamics. Some complex problems for multiphase flows and flows in porous media were solved by this method [1], [6].

One of the well-known computational algorithm of LBM is based on the splitting on physical processes [5], [8]. Splitting process is realized on every time step by two stages — collision and advection. On the advection step the solution of the system for the particle advection process is realized. The process is described by system of linear hyperbolic equations. On the collision stage nonlinear system without space derivatives is solved [15].

In this study finite difference scheme (FDS) for the solution of linear advection equation is considered. The scheme is dependent on dimensionless scalar parameter. The properties of the scheme such as stability and existence of fictitious effects (dispersion and dissipation) are investigated. The problems of the optimization of dispersive and dissipative surfaces are considered. Optimal parameter value is obtained.

The paper is organized as follows. In Section 2 the linear advection system and FDS are considered. In Section 3 the properties of FDS are investigated. Concluding remarks are made in Section 4.

2 FDS for linear advection system

The system of linear advection equations is written as:

$$\frac{\partial f_i}{\partial t} + V_i \nabla f_i = 0,$$

where $f_i = f_i(t, r)$, $i = 1, n$ are the distribution functions of the particles with velocities $V_i = V e_i$, $V = l/\delta t$, where $l$ is a mean free path, $\delta t$ is a mean free time, $t$ is a time, $r$ is a vector of space variables.

System (1) is solved on the advection stage of the method of splitting on physical processes in LBM or in solution procedure for general Boltzmann equation [4].

For the simplification of the analysis 1D variant of (1) is considered:

$$\frac{\partial f_i}{\partial t} + V_i \frac{\partial f_i}{\partial x} = 0,$$

where $i = 1, 2$, $e_1 = -1$, $e_2 = 1$ (D1Q2 lattice). Dimensionless variables are introduced:

$$t := \frac{t}{\delta t}, \quad x := \frac{x}{l}, \quad f_i := \frac{f_i}{\Phi_i},$$

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$$t := \frac{t}{\delta t}, \quad x := \frac{x}{l}, \quad f_i := \frac{f_i}{\Phi_i},$$
where $\Phi_i$ are the mean values of $f_i$.

After the substitution of (3) into (2) the following system is obtained:

$$\frac{\partial f_i}{\partial t} + e_i \frac{\partial f_i}{\partial x} = 0, \quad (4)$$

Equations of (4) are independent on each other, so the case of linear scalar equation may be considered:

$$\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0, \quad (5)$$

where $u = f_i$, $c = e_i$.

For the discretization of (5) the following formulae for time derivative is considered:

$$\frac{\partial u}{\partial t}(t_j, x_n) \approx \frac{u(t_{j+1}, x_n) - \frac{1}{2} (u(t_j, x_n) + u(t_{j-1}, x_n))}{2\Delta t},$$

where $u_n^j \approx u(t_j, x_n)$, where $\Delta t$ is a time step, $t_j$ is a time node, $x_n$ is a node of space grid constructed with step $h$.

Let us consider two approximations of the term $c \frac{\partial u}{\partial x}$ by central and first order upwind differences along the characteristics of eq. (5):

$$c \frac{\partial u}{\partial x}(t_j, x_n) \approx c \frac{u(t_j, x_n + \text{sign}(c)h) - u(t_j, x_n - \text{sign}(c)h))}{2h},$$

$$c \frac{\partial u}{\partial x}(t_j, x_n) \approx c \frac{u(t_j, x_n) - u(t_j, x_n - \text{sign}(c)h))}{h}.$$  

For the simplification of the analysis the case of $c > 0$ is considered. So the following two-step FDS’s for (5) are constructed:

$$u_{n+1}^j = \frac{1}{2} (u_n^j + u_{n-2}^j) - \gamma (u_{n+1}^j - u_{n-1}^j), \quad (6)$$

$$u_{n+1}^j = \frac{1}{2} (u_n^j + u_{n-2}^j) - 2\gamma (u_n^j - u_{n-1}^j), \quad (7)$$

where $\gamma = c\Delta t/h$ is a Courant number.

The stability of the schemes (6) and (7) may be analyzed by von Neumann method. The solutions of (6) and (7) are presented in following form:

$$u_n^j = \lambda^j(\varphi) e^{in\varphi}, \quad (8)$$

where $\varphi \in [0, 2\pi)$, $i^2 = -1$, $\lambda(\varphi)$ is a spectral function.

After the substitution of (8) into (6) and (7) the following cubic equations on $\lambda$ are obtained:

$$\lambda^3 + \left(i2\gamma \sin(\varphi) - \frac{1}{2}\right) \lambda^2 - \frac{1}{2} = 0,$$
\[ \lambda^3 + \left( i2\gamma(1 - e^{-i\varepsilon}) - \frac{1}{2} \right) \lambda^2 - \frac{1}{2} = 0. \]

The roots of these equations may be obtained by Cardano’s formulas or in packages of computer algebra. The following stability conditions are obtained after analysis of absolute values of the roots: \( \gamma \leq 1/2 \) for (6) and \( \gamma \leq 1/4 \) for (7).

The results of the solution of simple test Cauchy problem for the case of \( c = 1, x \in [0, 10] \) and initial condition: \( u(0, x) = 1 \) if \( x \in [4, 6] \) and \( u(0, x) = 0 \) elsewhere, are presented at fig. 1. As it can be seen, the fictitious numerical oscillations exists in the numerical solution obtained by scheme (6), this effect is known as a presence of numerical dispersion. Numerical solution obtained by (7) tend to null when \( t \to +\infty \), this effect is known as numerical dissipation.

![Fig. 1. Solutions of the Cauchy problem for eq. (5): 1 — numerical solution by FDS (6); 2 — analytical solution; 3 — numerical solution by FDS (7).](image)

To avoid these fictitious effects, the scalar parameter \( \varepsilon \in [0, 1] \) may be included to FDS. Due to the idea of Guo and Zhao [6], this parameter may be included by the following way:

\[
\frac{\partial u}{\partial x}(t_j, x_n) \approx \varepsilon \left( \frac{\partial u}{\partial x}(t_j, x_n) \right)_U + (1 - \varepsilon) \left( \frac{\partial u}{\partial x}(t_j, x_n) \right)_C,
\]

where \( (c\partial u(t_j, x_n)/\partial x)_U \) is approximation by first order upwind difference, \((c\partial u(t_j, x_n)/\partial x)_C \) is approximation by central difference.

The scheme obtained is presented by following formulae:

\[
u_{n+1}^j = \frac{1}{2} \left( u_n^j + u_{n-1}^j \right) - 2\gamma \varepsilon \left( u_n^j - u_{n-1}^j \right) - \gamma(1 - \varepsilon) \left( u_{n+1}^j - u_{n-1}^j \right).
\] (9)
3 Investigation of properties of the schemes

3.1 Stability

After the substitution of (8) into (9) the following cubic equation on $\lambda(\varphi)$ is obtained:

$$
\lambda^3 + \left(2\gamma(\varepsilon(1 - \cos(\varphi)) + i\sin(\varphi)) - \frac{1}{2}\right) \lambda^2 - \frac{1}{2} = 0. \tag{10}
$$

As a result of numerical analysis of roots of (10) stability domain of the scheme (9) in parameter space $(\varepsilon, \gamma)$ is constructed. At fig. 2 the plot of boundary of this domain is presented.

![Fig. 2. Boundary of the stability domain of scheme (9).](image)

3.2 Numerical dispersion and dissipation

Investigation of numerical dispersion and dissipation and optimization of FDS (9) is based on the analysis of dispersive and dissipative surfaces. The idea of this approach is presented in [7].

Solution of (9) is presented in form of traveling wave:

$$
u_n^j = e^{i(\omega\Delta t j - knh)}, \tag{11}
$$

where $\omega$ is a frequency, $k$ is a wave number.

After the substitution of (11) into (9), the following equation on $q = e^{i\omega\Delta t}$ is obtained:

$$
q^3 + \left(2\gamma(\varepsilon(1 - \cos(\xi)) - i\sin(\xi)) - \frac{1}{2}\right) q^2 - \frac{1}{2} = 0, \tag{12}
$$
where \( \xi = kh \). There are three roots of eq. (12): \( q_s, s = 1, 3 \). The real parts of frequencies \( \omega_s \), which are the main characteristics of numerical dispersion, may be expressed from \( q_s \) by following formulas:

\[
\psi_s(\gamma, \xi, \varepsilon) = \text{Re}(\omega_s) = \frac{\arctan \left( \frac{\text{Im}(q_s)}{\text{Re}(q_s)} \right)}{\Delta t}.
\] (13)

Dispersive relation for eq. (5) is presented as:

\[
\omega = ck.
\] (14)

Expressions for dispersive surface of eq. (5) may be obtained from (14) as a function of \( \gamma \) and \( \xi \):

\[
\omega(\gamma, \xi) = \frac{\gamma \xi}{\Delta t}.
\] (15)

Due to the dependence of \( \psi_s \) and \( \omega \) (see eqs. (13) and (15)) on \( \gamma \), the case of \( \Delta t = 1 \) may be considered.

Optimal dispersive properties of the scheme (9) may be characterized by following function:

\[
I(\varepsilon) = \sup_{s=1,2,3} \left( \sup_{(\gamma, \xi)} |\psi_s(\gamma, \xi, \varepsilon) - \omega(\gamma, \xi)| \right).
\]

Optimal value of parameter \( \varepsilon \) is considered as a solution of minimization problem of \( I(\varepsilon) \). This function is minimized on the domain \( \{(\gamma, \xi)|\gamma \in (0, \gamma^*(\varepsilon)), \xi \in [-\pi, \pi]\} \), where \( \gamma^*(\varepsilon) \) is an upper boundary of stability domain at fixed \( \varepsilon \).

For the characterization of dissipative properties eq. (11) is rewritten as:

\[
u_n^j = |q^j| e^{i(\text{Arg}(q^j) - kh)}.
\] (16)

As it can be seen from (16), dissipative property is characterized by functions \( \eta_s(\gamma, \xi, \varepsilon) = |q_s| \). For the obtaining of optimal value of \( \varepsilon \), the function \( F(\varepsilon) \), presented as:

\[
F(\varepsilon) = \sup_{s=1,2,3} \left( \sup_{(\gamma, \xi)} |\eta_s(\gamma, \xi, \varepsilon) - C_s| \right),
\] (17)

is minimized. In eq. (17) \( C_s \) are constants, which are the values of \( \eta_s \) in case of low dissipation (this case characterized by small values of \( \gamma \)).

Plots of \( I(\varepsilon) \) and \( F(\varepsilon) \) are presented at fig. 3 and fig. 4 respectively. As it can be seen, optimal parameter for \( I(\varepsilon) \) is equal to unity, while minimum of \( F(\varepsilon) \) is realized in internal point of interval \([0, 1]\) — in \( \varepsilon \approx 0.15 - 0.55 \). It is easy to concern (see fig. 1) that the main influence on numerical solution is realized by numerical dissipation, so the optimal parameter value is chosen to be equal to 0.15.
Fig. 3. Plot of $I(\varepsilon)$.

Fig. 4. Plot of $F(\varepsilon)$.

Fig. 5. Plot of numerical solution of the test Cauchy problem obtained by scheme (9) at optimal parameter value: 1 — analytical solution, 2 — numerical solution.
At fig. 5 the plot of numerical solution, obtained by scheme (9) with optimal parameter is presented. As it can be seen, numerical solution demonstrate the absence of fictitious oscillations. At the same time, the amplitude of the solution is close to constant and not decreased intensively.

4 Conclusion

FDS for linear advection equation with dependence on scalar dimensionless parameter is constructed. Stability of the scheme is investigated by von Neumann method and stability domain in parameter space is constructed. Dispersive and dissipative properties of the scheme are optimized by the choice of scalar parameter. Scheme constructed may be applied in computations based on splitting method for Boltzmann or lattice Boltzmann equations.

Acknowledgements. The reported study was funded by Russian Foundation of Basic Research according to research project No. 16-31-00021 mol_a.

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Received: May 15, 2016; Published: July 19, 2016