Existence of Nash Equilibria via Variational Inequalities in Riemannian Manifolds

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Abstract

In this paper, we first prove a generalization of McClendon’s variational inequality for contractible multimaps. Next, using a new generalized variational inequality, we will prove an existence theorem of Nash equilibrium for the generalized game \( \mathcal{G} = (X_i; T_i, f_i)_{i \in I} \) in a finite dimensional Riemannian manifold. A suitable example for Nash equilibrium is given in a geodesic convex generalized game.

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1 Introduction

Existence results of Nash equilibria are often derived from fixed point theorems, minimax inequalities, variational inequalities or KKM-type intersection theorems, e.g., see [3-7,10] and the references therein. Recently, there have been some progress on the existence of Nash equilibrium without using the convexity or even more, when the domains are not convex in the usual sense, e.g., [4,5] by embedding the strategy sets into suitable Riemannian manifolds.

In a very recent paper [4], by using a generalization of Begle’s fixed point theorem for compact acyclic multimaps, the author proves the existence theorem of Nash equilibrium for the generalized game \( \mathcal{G} = (X_i; T_i, f_i)_{i \in I} \) of normal
form with geodesic convex values in a finite dimensional Riemannian manifold which generalizes the existence theorem of a Nash equilibrium for the noncooperative game \( \mathcal{G} = (X_i; f_i)_{i \in I} \) due to Kristály [5].

In this paper, we first prove a generalization of McClendon’s variational inequality for contractible multmaps. Next, using a generalized variational inequality as a proving tool, we will prove the existence theorem of Nash equilibrium for the generalized game \( \mathcal{G} = (X_i; T_i, f_i)_{i \in I} \) of normal form with geodesic convex values in a finite dimensional Riemannian manifold. A suitable example for Nash equilibrium is given in a geodesic convex generalized game.

2 Preliminaries

We begin with some notations and definitions. If \( A \) is a nonempty set, we shall denote by \( 2^A \) the family of all subsets of \( A \). Let \( E \) be a topological vector space and \( X \) be a nonempty subset of \( E \). If \( T : X \to 2^E \) and \( S : X \to 2^E \) are multmaps (or correspondences), then \( S \cap T : X \to 2^E \) is a correspondence defined by \( (S \cap T)(x) = S(x) \cap T(x) \) for each \( x \in X \). When a multimap \( T : X \to 2^E \) is given, we shall denote \( T^{-1}(y) := \{ x \in X \mid y \in T(x) \} \) for each \( y \in E \).

Let \( I = \{1, 2, \ldots, n\} \) be a finite (or possibly countably infinite) set of players. For each \( i \in I \), \( X_i \) is a nonempty topological space as an action space, and denote \( X_{-i} := \Pi_{j \in I - \{i\}} X_j \). For an action profile \( x = (x_1, \ldots, x_n) \in X = \Pi_{i \in I} X_i \), we shall write \( x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in X_{-i} \), and we may simply write \( x = (x_{-i}, x_i) \in X_{-i} \times X_i = X \). A multimap \( T \) has open graph in \( X \) if the graph \( \text{Gr} T := \{(x, y) \mid y \in T(x) \text{ for each } x \in X\} \) is open in \( X \times E \).

When a multimap \( T_i : X \to 2^{X_i} \) has open graph in \( X \) for each \( i \in I \), and let \( T : X \to 2^X \) be a multimap defined by \( T(x) := \Pi_{i \in I} T_i(x) \) for each \( x \in X \). Then it is easy to see that the graph of \( T \) is open graph in \( X \times X \).

Recall that a topological space is contractible if the identity map is homotopic to a constant. A nonempty convex set is clearly contractible. A nonempty topological space is acyclic if all of its reduced Čech homology groups over rationals vanish. Every contractible space is acyclic, but the converse is not true, e.g., see [10]. Recall that a nonempty subset \( X \) of a Riemannian manifold \( M \) is said to be geodesic convex if every two points \( x, y \in X \) can be joined by a unique minimizing geodesic whose image belongs to \( X \). It is clear that the nonempty intersection of geodesic convex subsets of \( M \) is also a geodesic convex subset of \( M \).

Next, we recall some notions and terminologies on the generalized Nash equilibrium for pure strategic games as in [2-4,10]. Let \( I = \{1, 2, \ldots, n\} \) be a
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finite (or possibly countably infinite) set of players. A noncooperative generalized game of normal form is an ordered 3n-triple $G = (X_i; T_i, f_i)_{i \in I}$ where for each player $i \in I$, $X_i$ is a pure strategy space for the player $i$, and the set $X := \prod_{i=1}^{n}X_i = X_{-i} \times X_i$, joint strategy space, is the Cartesian product of the individual strategy spaces, and the element of $X_i$ is called a strategy. Let $f_i : \to \mathbb{R}$ be a payoff function (or utility function), and $T_i : X \to 2^{X_i}$ be a constraint correspondence for the player $i$. Then, a strategy $n$-tuple $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n) \in X$ is called a Nash equilibrium for the generalized game $G$ if for each $i \in I$,

$$\bar{x}_i \in T_i(\bar{x}) \quad \text{and} \quad f_i(\bar{x}_{-i}, \bar{x}_i) \leq f_i(\bar{x}_{-i}, y) \quad \text{for all} \quad y \in T_i(\bar{x}). \quad (\star)$$

In the game $G$, if $T_i(x) := X_i$ for each $x \in X$ and $i \in I$, then $\bar{x}_i \in T_i(\bar{x})$ in the equation $(\star)$ is trivial. In this case, the Nash equilibrium for the generalized game $G$ reduces to the Nash equilibrium in [8].

We shall need the following fixed point theorem which is the basic tool for proving the main result of this paper:

**Lemma 2.1.** [7] Let $X$ be a compact acyclic finite dimensional ANR, and $T : X \to 2^X$ has an open graph such that $T^{-1}(y)$ is nonempty contractible for each $y \in X$ (or, $T^{-1} : X \to 2^X$ has an open graph in $X \times X$ such that $T(x)$ is nonempty contractible for each $x \in X$). Then $T$ has a fixed point.

The following lemma is also essential for proving the main result:

**Lemma 2.2.** [5] Let $M$ be a complete finite dimensional Riemannian manifold. Then any geodesic convex subset $K$ of $M$ is contractible.

Next, we shall prove a generalization of McClendon’s variational inequality for contractible multimaps in [6] as follows:

**Lemma 2.3.** Suppose that $X$ is a compact acyclic finite dimensional ANR. Suppose $g : X \times X \to \mathbb{R}$ is a function and $T : X \to 2^X$ is a multimap such that

1. $\{(x, y) \in X \times X \mid g(x, x) > g(x, y)\}$ is open;
2. $T$ has open graph in $X \times X$, and $T(x)$ is nonempty for each $x \in X$;
3. $\{y \in X \mid g(x, x) > g(x, y)\} \cap T(x)$ is contractible or empty for each $x \in X$.

Then there is an $x_o \in X$ such that

$$g(x_o, x_o) \leq g(x_o, y) \quad \text{for each} \quad y \in T(x_o).$$

Furthermore, if $g(x_o, x_o) > g(x_o, y)$ for each $y \notin T(x_o)$, then

$$x_o \in T(x_o) \quad \text{and} \quad g(x_o, x_o) \leq g(x_o, y) \quad \text{for each} \quad y \in T(x_o).$$
Proof. First we define a multimap $S : X \rightarrow 2^X$ by
\[ S(x) := \{ y \in X \mid g(x, x) > g(x, y) \} \cap T(x) \quad \text{for each} \quad x \in X. \]

By the assumption (3), each $S(x)$ is contractible or empty. For each $y \in X$, we have
\[ S^{-1}(y) = \{ x \in X \mid y \in S(x) \} = \{ x \in X \mid g(x, x) > g(x, y) \} \cap T^{-1}(y). \]

By the assumptions (1) and (2), $T^{-1}$ has open graph so that $S^{-1}$ has an open graph in $X \times X$. Suppose that $S(x_o)$ is nonempty for each $x_o \in X$. Then, the multimap $S$ satisfies all the assumptions of Lemma 2.1 so that there exists a fixed point $\bar{y} \in X$ for $S$, i.e., $\bar{y} \in S(\bar{y})$. This implies that $g(\bar{y}, \bar{y}) > g(\bar{y}, \bar{y})$ which is impossible. Therefore, we have that $S(x_o)$ should be empty for some $x_o \in X$. Since $T(x_o)$ is nonempty, we can obtain the conclusion
\[ g(x_o, x_o) \leq g(x_o, y) \quad \text{for each} \quad y \in T(x_o). \]

Furthermore, by the assumption, if $x_o \notin T(x_o)$, then $g(x_o, x_o) > g(x_o, x_o)$ which is a contradiction. Therefore, we obtain that $x_o \in T(x_o)$ which completes the proof. \hfill \Box

In Lemma 2.3, when $T(x) := X$ for each $x \in X$, then the assumption (2) is automatically satisfied so that McClendon’s variational inequality in [6, Theorem 3.1] is obtained as a corollary.

Throughout this paper, all spaces are assumed to be a complete finite dimensional Riemannian manifold, and for some basic definitions and standard terminologies on Riemannian geometry, we shall refer to [5-7,9,11], and the references therein.

3 Existence of a generalized Nash equilibrium

Recall that when $X$ is a nonempty geodesic convex subset of a finite dimensional Riemannian manifold $M$, then $f : X \rightarrow \mathbb{R}$ is called a convex function in $X$ if $f \circ \gamma : [0, 1] \rightarrow \mathbb{R}$ is convex in the usual sense for every geodesic $\gamma : [0, 1] \rightarrow X$.

We shall prove the following existence theorem of Nash equilibrium for a generalized game $\mathcal{G} = (X_i; T_i, f_i)_{i \in I}$ in compact geodesic convex subsets of finite dimensional Riemannian manifolds $M_i$:

Theorem 3.1. Let $I$ be a finite set of players, and let $\mathcal{G} = (X_i; T_i, f_i)_{i \in I}$ be a noncooperative generalized game of normal form, where $X_i$ is a nonempty compact geodesic convex ANR for the player $i$ in a finite dimensional Riemannian manifold $M_i$. Assume that the joint strategy space $X = \Pi_{i \in I} X_i$ is a
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subset of $M = \Pi_{i \in I} M_i$. Suppose that for each $i \in I$, $f_i : X_{-i} \times X_i \to \mathbb{R}$ is a function, and $T_i : X \to 2^X$ has open graph in $X \times X$, satisfying the following conditions: for each $i \in I$, and $x = (x_{-i}, x_i) \in X$,

(1) $f_i : X_{-i} \times X_i \to \mathbb{R}$ is (jointly) continuous in $X$;
(2) $f_i : X_{-i} \times X_i \to \mathbb{R}$ is convex in $X_i$;
(3) $T_i(x)$ is nonempty geodesic convex;
(4) $T$ satisfies the reflexive property, i.e., $x_i \in T_i(x)$ for each $x \in X$.

Then there exists an equilibrium point $\bar{x} \in X$ for the game $G = (X_i; T_i, f_i)_{i \in I}$ such that for each $i \in I$,

$$\bar{x}_i \in T_i(\bar{x}) \quad \text{and} \quad f_i(\bar{x}_{-i}, \bar{x}_i) \leq f_i(\bar{x}_{-i}, y_i) \quad \text{for all} \quad y_i \in T_i(\bar{x}).$$

**Proof.** Let $X = \Pi_{i \in I} X_i = X_{-i} \times X_i$, and $f : X \times X \to \mathbb{R}$ and $T : X \to 2^X$ be defined by

$$f(x, y) := \sum_{i \in I} (f_i(x_{-i}, y_i) - f_i(x_{-i}, x_i)) \quad \text{for each} \quad (x, y) \in X \times X;$$

$$T(x) := \prod_{i \in I} T_i(x) \quad \text{for each} \quad x \in X.$$

First, we note that since a product of a finite family of ANRs is an ANR (e.g., see [1, Corollary 5.5]), $X$ is a compact geodesic convex finite dimensional ANR. By Lemma 2.2, $X$ is contractible so that $X$ is an acyclic ANR.

Note that since the function $f : X \times X \to \mathbb{R}$ is continuous by the assumption (1) and $f(x, x) = 0$, the set $\{(x, y) \in X \times X \mid 0 = f(x, x) > f(x, y)\}$ is open. Since each $T_i$ has open graph in $X \times X_i$, the multimap $T$ has open graph in $X \times X$.

Next, we shall prove that the set $U := \{y \in X \mid 0 > f(x, y)\} \cap T(x)$ is contractible or empty for each $x \in X$. First assume that $U$ is nonempty. Then it suffices to show that $U$ is contractible. Since $U$ is nonempty, there exists an $y \in T(x)$ such that $0 > f(x, y)$, i.e., there exists some $i \in I$ with $y_i \in T_i(x)$ such that $f_i(x_{-i}, y_i) - f_i(x_{-i}, x_i) < 0$. By the assumption (4), $x_j \in T_j(x)$ for each $x \in X$ and $j \in I$, and hence it is easy to see that $(x_{-i}, y_i) \in U$.

Now we fix $y^j = (y^j_1, \ldots, y^j_n) \in U$ for each $j = 1, 2$. Then, for each $i \in I$, $y^1_i, y^2_i \in T_i(x)$. Since each $T_i(x)$ is geodesic convex, let $\gamma_i : [0, 1] \to X_i$ be the unique geodesic joining the points $y^1_i = \gamma_i(0), y^2_i = \gamma_i(1)$ contained in $T_i(x)$. Next we define $\gamma : [0, 1] \to X$ by

$$\gamma(t) := (\gamma_1(t), \ldots, \gamma_n(t)) \quad \text{for all} \quad t \in [0, 1].$$
By the convex assumption (2) on $f_i$, for every $t \in [0, 1]$, we have
\[
    f(x, \gamma(t)) = \sum_{i \in I} \left( f_i(x_{-i}, \gamma_i(t)) - f_i(x_{-i}, x_i) \right)
    \leq \sum_{i \in I} \left( tf_i(x_{-i}, \gamma_i(0)) + (1-t)f_i(x_{-i}, \gamma_i(1)) - f_i(x_{-i}, x_i) \right)
    = tf(x, y^1) + (1-t)f(x, y^2) < 0.
\]

Consequently, we have $\gamma(t) \in \{ y \in X \mid 0 > f(x, y) \}$ for every $t \in [0, 1]$ so that it is geodesic convex. Since $T_i(x)$ is a nonempty geodesic convex set in the product manifold $M = \prod_{i \in I} M_i$ endowed with its natural product metric so that each $T(x)$ is geodesic convex. Therefore, $U = \{ y \in X \mid 0 > f(x, y) \} \cap T(x)$ is also geodesic convex and hence, by Lemma 2.2 again, $U$ is contractible.

Therefore, all the hypotheses of Lemma 2.3 are satisfied so that there exists a point $\bar{x} \in X$ such that
\[
    0 = f(\bar{x}, \bar{x}) \leq f(\bar{x}, y) \quad \text{for all } y \in T(\bar{x}).
\]

In the inequality (*), for each $i \in I$, if we take any $y_i \in T_i(\bar{x})$, then by the reflexive assumption (4), $y = (\bar{x}_{-i}, y_i) \in T(\bar{x})$ so that we have
\[
    f_i(\bar{x} - x, \bar{x}) \leq f_i(\bar{x}_{-i}, y_i) \quad \text{for all } y_i \in T_i(\bar{x}).
\]

Then, by the assumption (4) again, we conclude that $\bar{x} \in X$ is actually an equilibrium point for the game $G = (X_i; T_i, f_i)_{i \in I}$. Indeed, for each $i \in I$,
\[
    \bar{x}_i \in T_i(\bar{x}) \quad \text{and} \quad f_i(\bar{x} - x, \bar{x}_i) \leq f_i(\bar{x}_{-i}, y_i) \quad \text{for all } y_i \in T_i(\bar{x}),
\]
which completes the proof. $\square$

**Remarks.** (1) Theorem 3.1 is very different from Theorem 1 in [3,5,8,10] in the following aspects:
(a) we do not need the convex assumption on $X_i$;
(b) we do not need the (usual) convex assumption on $f_i$;
(b) we do not need the convex assumption on $T_i(x)$.

In Theorem 3.1, when $T_i(x) := X_i$ for each $i \in I$ and $x \in X$, the the assumptions (3), (4) and the open graph assumption on $T_i$ are automatically satisfied so that as a consequence of Theorem 3.1, we can obtain the existence theorem of a Nash equilibrium for the game $G = (X_i; T_i, f_i)_{i \in I}$ due to Kristály [5]:

**Theorem 3.2.** Let $I$ be a finite set of players, and let $G = (X_i; f_i)_{i \in I}$ be a noncooperative game where $X_i$ is a nonempty compact geodesic convex strategy subset of the player $i$ in a finite dimensional Riemannian manifold $M_i$. Assume that for each $i \in I$, 

\[ ... \]
Example 3.3. Let \( G = (X_i; T_i, f_i)_{i \in I} \) be a nonconvex generalized game such that the pure strategic space \( X_i \) for each player \( i \) is defined by

\[
X_1 := \{(x_1, x_2) \in \mathbb{R}^2 \mid -1 \leq x_1, x_2 \leq 1\};
\]

\[
X_2 := \{(\cos t, \sin t) \in \mathbb{R}^2 \mid \frac{\pi}{4} \leq t \leq \frac{3\pi}{4}\}.
\]

Then, \( X_1 \) is a compact convex subset of \( \mathbb{R}^2 \) in the usual sense, and \( X_2 \) is compact but not a convex subset of \( \mathbb{R}^2 \) in the usual sense. However, as remarked in [5], if we consider the Poincaré upper-plane model \((\mathbb{H}^2, g_H)\), then the set \( X_2 \) is geodesic convex with respect to the metric \( g_H \) being the image of a geodesic segment from \((\mathbb{H}^2, g_H)\).

For each player \( i = 1, 2 \), let the payoff function \( f_i : X = X_1 \times X_2 \to \mathbb{R} \), and a continuous constraint correspondence \( T_i : X \to 2^X \) are defined as follows: for each \((x_1, x_2), (y_1, y_2)\) \( \in X = X_1 \times X_2\),

\[
f_1((x_1, x_2), (y_1, y_2)) := |x_1| y_1^2 - y_2,
\]

\[
f_2((x_1, x_2), (y_1, y_2)) := (1 - |x_2|)(y_1^2 - y_2^2);
\]

\[
T_1((x_1, x_2), (y_1, y_2)) := X_1, \quad T_2((x_1, x_2), (y_1, y_2)) := X_2.
\]

Then the action sets \( X_i \) are compact and geodesic convex, and both payoff functions \( f_i \) are continuous. It is easy to see that \( f_1(\cdot, (y_1, y_2)) \) is convex on \( X_1 \). Since the function \( t \mapsto (1 - |x_2|) \cos 2t \) is a convex function on \([\frac{\pi}{4}, \frac{3\pi}{4}]\), \( f_2((x_1, x_2), \cdot) \) is clearly a convex function on \( X_2 \).

For each \( i \in I \), \( T_i \) has open graph in \( X \times X_i \) such that each \( T_i(x, y) = X_i \) is geodesic convex, and the reflexive property of \( T_i \) is trivial. Therefore, we can apply Theorem 3.1 to the generalized game \( G \), then we can obtain an equilibrium point \((0, 0), (0, 1)\) \( \in X = X_1 \times X_2 \) for \( G \) such that \((0, 0) \in T_1((0, 0), (0, 1)); \ (0, 1) \in T_2(((0, 0), (0, 1)) \); and

\[
-1 = f_1((0, 0), (0, 1)) \leq f_1((x_1, x_2), (0, 1)) \quad \text{for all} \ (x_1, x_2) \in T_1(x, y) = X_1,
\]

\[
-1 = f_2((0, 0), (0, 1)) \leq f_2((0, 0), (y_1, y_2)) \quad \text{for all} \ (y_1, y_2) \in T_2(x, y) = X_2.
\]

However, we can see that the previous equilibrium existence theorems due to Ding et al. [3], Kristály [5], and Park [10] for compact convex games can not be applicable to this geodesic convex game \( G \).
4 Conclusion

In this paper, using a generalization of McClendon’s variational inequality for contractible multimaps, we prove an existence theorem of Nash equilibrium for the compact geodesic convex generalized game \( G = (X_i; T_i; f_i)_{i \in I} \) of normal form with geodesic convex values in a finite dimensional Riemannian manifold.

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