On the Non-Differentiability of the Solutions of the

Fractional Differential Equation $y^{(\alpha)}(x) = \lambda y(x)$

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E pur si muove!

Galileo

Abstract

It is by-now customarily taken for granted that the solution of the fractional differential equation $D^\alpha y(x) = \lambda y(x)$ is the Mittag-Leffler function $E_\alpha(\lambda x^\alpha)$ which is differentiable everywhere but at the origin $x=0$. Here one shows that if one considers fractional derivative via fractional difference, then this equation might have solutions which are nowhere differentiable. Various formulae are derived with respect to this question, and various open problems are outlined. Relation with log-self-similarity is exhibited. The study as the whole is based on the differential form of the equation on the one hand, and the fractional Taylor series which has been derived by the author a few time ago for non-differentiable functions on the other hand.

Keywords: Fractional differential equation

1. Introduction

It is by now taken for granted that the solution of the fractional differential equation (referred to as equation $E$ in the following)

$$y^{(\alpha)}(x) = \lambda y(x), \quad \lambda \in R, \quad 0 < \alpha < 1 \quad , \quad y(0) = 1 \quad (1.1)$$
is the Mittag-Leffler function $E_\alpha(\lambda x^\alpha)$ of which the definition is (the symbol := means that the left side is defined by the right one)

$$E_\alpha(x) := \sum_{k=0}^{\infty} \frac{x^k}{(\alpha k)!},$$  \hspace{1cm} (1.2)

with the notation

$$(\alpha k)! := \Gamma(1 + \alpha k).$$  \hspace{1cm} (1.3)

where $\Gamma(x)$ is the Eulerian function.

And in effect, it is easy to check that the fractional derivative of (1.2) complies with equation $E$. This solution $y(x) = E_\alpha(\lambda x^\alpha)$ is continuous everywhere on the one hand, and differentiable everywhere but at $x = 0$ where it has a fractional derivative of order $\alpha$, on the other hand. Our purpose in the following is to show that if we consider the solution of the equation $E$ in the framework of fractional calculus via fractional difference, then this equation $E$ may have solutions which are everywhere non-differentiable. In addition we shall come across a new concept of log-self-similarity.

### 2. Summary of fractional calculus via fractional difference

Some preliminary concepts of fractional calculus are borne in mind in this section.

**Definition 2.1** Suppose that $f: \mathbb{R} \to \mathbb{R}, x \to f(x)$, is a continuous (but not necessarily differentiable) function, and let $h > 0$ denote a constant discretization span. Then the fractional difference of order $\alpha \in \mathbb{R}, \ 0 < \alpha \leq 1$ of $f(x)$ is defined as follows,

$$\Delta^\alpha f(x) = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f[x + (\alpha - k)h].$$  \hspace{1cm} (2.1)

**Definition 2.2** (modified Riemann-Liouville derivative). Refer to the function referred to in Definition 2.1, its fractional derivative of order $\alpha \in \mathbb{R}, \ 0 < \alpha \leq 1$ of $f(x)$ is defined as follows (for more details see [20])

$$f^{(\alpha)}(x) = \frac{d^\alpha f(x)}{dx^\alpha} := \frac{1}{\Gamma(-\alpha)} \int_0^x (x - \xi)^{-\alpha-1} (f(\xi) - f(0)) d\xi, \quad \alpha < 0$$

$$= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x - \xi)^{-\alpha} (f(\xi) - f(0)) d\xi, \quad 0 < \alpha < 1$$  \hspace{0.8cm} (2.2)

$$= \left(f^{(\alpha-n)}(x)^{(n)}\right), \quad n < \alpha \leq n + 1.$$

**Proposition 2.1** Refer to the function in Definition 2.1, then the following equality holds, that is
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\[ f^{(\alpha)}(x) = \lim_{h \to 0} \frac{\Delta^\alpha f(x)}{h^\alpha}. \]  

**Proposition 2.2** Assume that the continuous function \( f: \mathbb{R} \to \mathbb{R}, x \to f(x) \) has fractional derivative of fractional order \( k\alpha = \alpha + \alpha + \cdots + \alpha \) for any positive integer \( k \) and any \( \alpha, 0 < \alpha \leq 1 \). Then the following fractional Taylor's series holds, which is

\[ f(x + h) = \sum_{k=0}^{\infty} \frac{h^{k\alpha}}{\Gamma(1+k\alpha)} f^{(k\alpha)}(x). \]  

where \( f^{(k\alpha)}(x) \) is the derivative of order \( k\alpha \) of \( f(x) \) and \( \Gamma \) is the Gamma function.

**Proof.** For the first proof, see [16,20]. An alternative is to check that the fractional Taylor series applies to the Mittag-Leffler function, and then to consider functions which can be approximated by sequences of Mittag-Leffler functions.

**Corollary 2.1** Under the conditions in Proposition 2.2, for \( 0 < \alpha \leq 1 \), we have the equality

\[ d^\alpha f(x) = \Gamma(1 + \alpha)df(x) = \alpha! df(x). \]  

which holds for non-differentiable functions only. Remark that one has as well

\[ \Delta^\alpha f(x) = \alpha! \Delta f(x). \]  

**Proof.** From the equation (2.4) one obtains

\[ f(x + h) = f(x) + \frac{h^{\alpha}}{\Gamma(1+\alpha)} f^{(\alpha)}(x) + \sum_{k=2}^{\infty} \frac{h^{k\alpha}}{\Gamma(1+k\alpha)} f^{(k\alpha)}(x). \]  

This being the case, according to the definition of forward operator of order one, we have

\[ \Delta f(x) = \frac{h^{\alpha}}{\alpha!} f^{(\alpha)}(x) + \sum_{k=2}^{\infty} \frac{h^{k\alpha}}{(\alpha k)!} f^{(k\alpha)}(x). \]  

Multiplying both sides of (2.7) by \( \alpha! / h^{\alpha} \) and taking the limit as \( h \to 0 \) yields

\[ \alpha! \lim_{h \to 0} \frac{\Delta f(x)}{h^{\alpha}} = f^{(\alpha)}(x). \]  

Therefore, according to (2.3) and (2.8), we obtain

\[ \Delta^\alpha f(x) = \alpha! \Delta f(x), \]
or in the differential form
\[ d^\alpha f(x) = \alpha! \, df(x). \]  \hfill (2.10)

**Corollary 2.2** Relation between \( dx^\alpha \) and \( dx \). By assuming that \( \gamma > 0 \), the following equality holds, which is  \[ D^\alpha (x^\gamma) = (\gamma!/(\gamma - \alpha)!)x^{\gamma-\alpha}. \quad \gamma > 0. \]  \hfill (2.11)

For \( \gamma = 1 \), Eq. (2.11) turns to be  \[ d^\alpha x = 1![(1 - \alpha)!]^{-1}x^{1-\alpha}dx^\alpha = \frac{1}{(1-\alpha)!}x^{1-\alpha}dx^\alpha, \]  \hfill (2.12)

therefore, according to the equation (2.5)  \[ dx^\alpha = \alpha! (1 - \alpha)!x^{\alpha-1}dx. \]  \hfill (2.13)

**Corollary 2.3** Fractional Leibniz chain rule for non-differentiable functions

Assume that \( f(x) \) and \( g(x) \) both are non-differentiable when \( x = a \), but are \( \alpha \)-differentiable, \( 0 < \alpha < 1 \). Then the following equality holds, that is  \[ (f(x)g(x))^{(\alpha)}_{x=a} = f^{(\alpha)}(x)|_{x=a}g(a) + f(a)g^{(\alpha)}(x)|_{x=a}. \]  \hfill (2.14)

**Proof.** The proof is based on the fact that, as the functions which are dealt with are non-differentiable, according to their fractional Taylor’s series one can write  \[ \Delta f(x) = (\alpha!)^{-1}f^{(\alpha)}(x)|_{x=a}h^\alpha + o(h^\alpha), \]  \hfill (2.15)

therefore  \[ \Delta^\alpha f(x) = f^{(\alpha)}(x)|_{x=a}h^\alpha + o(h^\alpha), \]  \hfill (2.16)

It is then sufficient to write the equality  \[ \Delta(fg) = f(\Delta g) + g(\Delta f), \]  \hfill (2.17)

to multiply both sides by \( \alpha! \), and then to divide both sides by \( h^\alpha \) to get the result (by letting \( h \) tends to zero). This equality (2.14) will be of paramount importance in the sought of solution to the equation \( E \). Remark that (2.14) applies only when both \( f(x) \) and \( g(x) \) are not differentiable.
Corollary 2.4 Assume that \( u(x) \) is a differentiable function when \( x = a \). Assume further that \( f(x) \) is a non-differentiable function when \( x = a \), which is furthermore fractional differentiable of order, \( 0 < \alpha < 1 \), when \( x = a \). Then the following equality holds which reads

\[
D^\alpha(u(x)f(x))_{x=a} = u(a)f^{(\alpha)}(x)|_{x=a} \tag{2.18}
\]

Loosely speaking, everything happens as if \( u(x) \) were a constant with respect to \( f(x) \).

**Proof** Similar to the proof of Corollary 2.3, but here one has

\[
\Delta u = u'(x) \, dx
\]

while \( \Delta f \) is still given by (2.15).

**Illustrative example**

As an illustrative example of application, let us calculate the derivative of \( f(u(x)) \) where \( f(.) \) and \( u(.) \) are non-differentiable.

(Step 1) One first remark that

\[
d^\alpha x = \frac{x^{1-\alpha}}{(1-\alpha)!} \, dx^{\alpha}, \tag{2.19}
\]

or in a like manner

\[
dx^{\alpha} = (1 - \alpha)! \, x^{\alpha-1} d^\alpha x. \tag{2.20}
\]

(Step 2) This being the case one has

\[
\frac{d^\alpha f(u)}{dx^\alpha} = \frac{d^\alpha f}{du^\alpha} \left( \frac{du}{dx} \right)^\alpha \tag{2.21}
\]

\[
= \frac{d^\alpha f}{du^\alpha} (1 - \alpha)! \, u^{\alpha-1} \frac{d^\alpha u}{dx^\alpha}
\]

\[
= \frac{d^\alpha f}{du^\alpha} \frac{d^\alpha u}{dx^\alpha} (1 - \alpha)! \, u^{\alpha-1}. \tag{2.22}
\]

Strictly speaking, the derivative in (2.21) does not exist since \( u(x) \) is not differentiable. So, it is only a formal derivative to be converted into fractional derivative, as in equation (2.22).

3. On the non-differentiability of the solution of the equation \( E \)

**Lemma 3.1** Considered in the framework of the definition (2.1) for fractional derivative, the solution of the fractional differential equation (1.1) is nowhere differentiable provided that the fractional derivative so involved is different from zero.


**Proof** According to the fractional Taylor series of first order with respect to $\alpha$, one has the differential

$$dy = (\alpha!)^{-1} y^{(\alpha)}(x)(dx)^{\alpha} \quad (3.1)$$

therefore the “derivative”

$$y'(x) = \frac{dy}{dx} = (\alpha!)^{-1} \frac{y^{(\alpha)}(x)}{(dx)^{1-\alpha}}. \quad (3.2)$$

It follows that, according to (3.1) and (3.2) the solution of (1.1) is everywhere continuous but nowhere differentiable.

An alternative is to proceed as follows.

Let us assume that $y(x)$ is a solution for $x = x_0$, i.e.

$$y^{(\alpha)}(x_0) \bigg|_{x=x_0} = \lambda y(x_0), \quad (3.5)$$

and let us seek a solution which, in the neighborhood of $x_0$, has a fractional Taylor’s series in the form

$$y(x) = y(x_0) + \frac{1}{\alpha!} y^{(\alpha)}(x_0)(x - x_0)^{\alpha} + o((x - x_0)^{2\alpha}). \quad (3.6)$$

Then on taking account of *equation E*, one can substitute $\lambda y(x_0)$ for $y^{(\alpha)}(x_0)$ in this equality, to straightforwardly obtain

$$y(x) = y(x_0) + \frac{\lambda}{\alpha!} y(x_0)(x - x_0)^{\alpha} + o((x - x_0)^{2\alpha}), \quad (3.7)$$

therefore

$$\frac{dy}{dx} = \frac{\lambda}{\alpha!} y(x_0)(x - x_0)^{\alpha-1} + o((x - x_0)^{2\alpha-1}). \quad (3.8)$$

Remark that (3.2) can be re-written in the form

$$\alpha! y'(x) dx = y^{(\alpha)}(x)(dx)^{\alpha}$$

therefore the equality

$$\alpha! \int y'(x) dx = \int y^{(\alpha)}(dx)^{\alpha}.$$  

4. **Relation between *equation E* and Mittag-Leffler function**

4.2 Mittag-Leffler function and fractional Taylor’s series

It is by now usually taken for granted that the solution of the equation
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\[ y^{(a)}(x) = y(x), \quad y(0) = 1 \]  
(4.1)

is the Mittag-Leffler function \( E_{(a)}(x^a) \). Let us examine more closely this relation.

According to the fractional Taylor’s series (2.4), if \( y(x) \) is non-differentiable but is \( \alpha - th \) differentiable, then one can expand it in the form

\[ y(x + h) = y(x) + \sum_{k=1}^{\infty} \frac{1}{(k\alpha)!} y^{(a_k)}(x) h^{\alpha_k}. \]  
(4.2)

This being the case, according to the equation (4.1) one has the equality

\[ y^{(k\alpha)}(x) = y(x) \]  
(4.3)

in such a manner that (4.2) turns to be

\[ y(x + h) = y(x) \sum_{k=1}^{\infty} \frac{1}{(k\alpha)!} h^{\alpha_k}, \]  
(4.4)

that is to say

\[ y(x + h) = y(x)E_{\alpha}(h^\alpha). \]  
(4.5)

By using properties of symmetry, we arrive at the equation

\[ y(x + h) = y(x)y(h). \]  
(4.6)

4.2 On a pitfall involved in non-differentiability

If we set \( x = 0, h \leftarrow x \) in (4.5), we get

\[ y(x) = E_{\alpha}(x^\alpha), \]  
(4.7)

and we arrive at the equation

\[ E_{\alpha}((x + y)^\alpha) = E_{\alpha}(x^\alpha)E_{\alpha}(y^\alpha), \]  
(4.8)

which is amazing at first glance.

One can verify easily, with a simple counterexample, that the equation (4.7) does not hold in a standard manner. It is satisfied when \( x = y = 0 \), but otherwise, what happens?

As a matter of fact, we start from the Taylor series (4.2) which applies only in a point \( x \) where \( y(x) \) is not differentiable. In other words, the equation (4.6) which is

\[ y(u + v) = y(u)y(v) \]

applies only for some \( u \) and \( v \) where \( y(u) \) and \( y(v) \) are non-differentiable. In other words, the equation (4.8) holds only when \( x = 0 \) and \( y = 0 \). Or again, (4.6) applies only when \( y(x) \) is nowhere differentiable.
5. On the differential form of the equation \( E \)

5.1 Standard solution of the equation \( E \)

Let us bear in mind that it is by-now standard to take for granted that the solution of the equation \( E \) is the Mittag-Lefler function and this can be checked easily. Indeed, starting from the fractional derivative (2.11) of \( x^\gamma \), (1.2) direct yields (do not forget that the fractional derivative of a constant is zero)

\[
D^\alpha E_\alpha (x^\alpha) = \sum_{k=1}^{\infty} \frac{d^\alpha x^k}{(ak)!} \\
= \sum_{k=1}^{\infty} \frac{(ak)!}{(ak)!((ak-\alpha)!)} x^{\alpha k - \alpha} \\
= \sum_{k=1}^{\infty} \frac{1}{(ak)!} x^{\alpha k}
\]

For small \( x \), one has

\[
E_\alpha (x^\alpha) = 1 + \frac{x^\alpha}{\alpha!} + o(x^{2\alpha})
\]
in such a manner that \( E_\alpha (x^\alpha) \) appears to be differentiable everywhere but at \( x = 0 \).

And in quite a natural way, the question which comes in mind is as to whether the equation \( E \) has a solution which is not differentiable.

5.2 Differential form of the equation \( E \)

By using the definition of fractional derivative via fractional difference, one can directly translate the model

\[
y^{(\alpha)}(x) = \lambda y(x),
\]
in the form

\[
d^\alpha y = \lambda y(dx)^\alpha.
\]

Before to go on, we remark that if \( y(x) \) is a solution of the equation \( E \), then one has as well

\[
y^{(k\alpha)}(x) = \lambda^k y(x).
\]

This being the case, at a point \( x \) where the function \( y(x) \) is not differentiable, the fractional Taylor's series yields successively

\[
y(x + dx) = y(x) + \sum_{k=1}^{\infty} \frac{1}{(k\alpha)!} y^{(k\alpha)}(x)(dx)^k\alpha,
\]

\[
y(x + dx) = y(x) + \sum_{k=1}^{\infty} \frac{1}{(k\alpha)!} \lambda^k y(x)(dx)^k\alpha,
\]

\[
y(x + dx) = y(x) \left[ 1 + \sum_{k=1}^{\infty} \frac{\lambda^k}{(k\alpha)!}(dx)^k\alpha \right].
\]
therefore the new differential equation
\[ y(x + dx) = y(x)E_\alpha(\lambda(dx)^\alpha). \] (5.7)
which provides the approximation (for small \( h \))
\[ y(x + h) \approx y(x)(1 + \lambda(\alpha!)^{-1}h^\alpha). \] (5.8)

It follows that, in the vicinity of \( x \), the function \( y(x + h) \) is a non-differentiable function with respect to \( h \).

### 5.3 Verification of the differential form of equation \( E \)

**First verification**

As a verification, one can check that the equation (5.7) provides (5.1) and to this end we shall use the equality (2.10) which we bear in mind, (and which works for non-differentiable functions)
\[ d^\alpha y(x) = \alpha! \, dy(x). \] (5.9)
Equation (5.7) provides (5.8) which we re-write as
\[ \alpha! [y(x + dx) - y(x)] = \lambda y(x)(dx)^\alpha, \]
that is to say
\[ d^\alpha y(x) = \lambda y(x)(dx)^\alpha \] (5.10)
which is exactly (5.1).

In the special case of the differential equation (that is to say that when \( \alpha = 1 \))
\[ y'(x) = \lambda y(x) \]
one obtains the equivalent form
\[ y(x + dx) = y(x)e^{\lambda dx}. \]

**Second verification**

Another approach is as follows. One first notices that, at the point \( x \) where \( y(x) \) is non-differentiable, one has successively
\[ y(x + h) = y(x)E_\alpha(\lambda h^\alpha) \]
\[ = y(h + x) \]
\[ = y(h)E_\alpha(\lambda x^\alpha), \] (5.12)
from where we obtain
\[ D^\alpha_\delta y(x + h) = y(x)\left(\lambda E_\alpha(\lambda h^\alpha)\right) \] (5.13)
and
\[ D^\alpha_\delta y(h + x) = y(h)\left(\lambda E_\alpha(\lambda x^\alpha)\right), \] (5.14)
therefore the equality
\[ y(x)\left(\lambda E_\alpha(\lambda h^\alpha)\right) = y(h)\left(\lambda E_\alpha(\lambda x^\alpha)\right), \] (5.15)
which provides
\[ \frac{y(x)}{E_\alpha(\lambda x^\alpha)} = \frac{y(h)}{E_\alpha(\lambda h^\alpha)} = \mu, \] (5.16)
where \( \mu \) denotes a constant which is necessarily the unit.
6. Functional equation associated with the equation $E$

(Step 1) We start from the equation (5.11) which we recall below for convenience

$$y(x + h) = y(x)E_\alpha(\lambda h^\alpha),$$

(6.1)

and which holds for small $h$ and under two conditions: firstly, $y(x)$ is not differentiable at $x$, and secondly, $h$ is small enough to ensure the convergence of the fractional Taylor series so involved.

(Step 2) This being the case, let us consider the amplitude of $y(x + 2h)$ where $h$ denotes a small increment. Equation (6.1) directly yields

$$y(x + 2h) = y(x)E_\alpha(\lambda (2h)^\alpha),$$

(6.2)

but instead, we shall refer to the approximation

$$y(x + 2h) = y((x + h) + h)
= y(x + h)E_\alpha(\lambda h^\alpha)
= y(x)E_\alpha^2(\lambda h^\alpha),$$

therefore (recursively) the equality

$$y(x + kh) = y(x)(E_\alpha(\lambda h^\alpha))^k,$$

(6.3)

where $k$ denotes a positive integer.

(Step 3) We generalize this equation in the form

$$y(x + \rho h) = y(x)(E_\alpha(\lambda h^\alpha))^\rho,$$

(6.4)

for any positive $\rho$.

(Step 4) We now come back to the estimate of $y(x + h)$ that, for convenience, we re-write $y(x + u)$ to yield

$$y(x + u) = y\left(x + \rho \left(\frac{u}{\rho}\right)\right)
= y(x)\left(E_\alpha\left(\lambda \left(\frac{u}{\rho}\right)^\alpha\right)\right)^\rho.$$  

(6.5)

(Step 5) In order to convert the Mittag-Leffler term, we shall come back to the standard solution of the equation $E$, which reads (recall that $y(0) = 1$)

$$y(x) = E_\alpha(\lambda x^\alpha)$$

and provides

$$\left(E_\alpha\left(\lambda \left(\frac{u}{\rho}\right)^\alpha\right)\right)^\rho = y^\rho \left(\frac{u}{\rho}\right);$$
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and on substituting into (5.21) we obtain the functional equation

\[ y(x + u) = y(x) \left( y \left( \frac{u}{\rho} \right) \right)^\rho \]  

(6.6)

which yields

\[ y(u) = y(0) \left( y \left( \frac{u}{\rho} \right) \right)^\rho. \]  

(6.7)

(Step 6) Recall that, according to the definition of the equation \( E \), one has the equality \( y(0) = 1 \), in such a manner that (6.7) direct yields (we switch from \( u \) to \( x \ ))

\[ y(x) = \left( y \left( \frac{x}{\rho} \right) \right)^\rho \]  

(6.8)

which provides

\[ y(0) = (y(0))^\rho \]

that is to say

\[ y(0) = 1. \]  

(Step 7) Let us proceed to the checking of the equation (6.5), i.e.

\[ y(x + h) = y(x) \left( E_\alpha \left( \lambda \left( \frac{h}{\rho} \right) \right)^\alpha \right). \]  

(6.9)

One has successively

\[ \frac{d^\alpha}{dh^\alpha} \left[ E_\alpha \left( \lambda \frac{h^\alpha}{\rho^\alpha} \right)^\rho \right] = \frac{d^\alpha (e_\alpha^\rho)}{dE_\alpha^\alpha} \frac{(dE_\alpha)^\alpha}{(dE_\alpha^\alpha)^\alpha} \frac{d^\alpha E_\alpha}{dE_\alpha^\alpha} \frac{d^\alpha h}{dE_\alpha^\alpha}, \]  

(6.10)

\[ \frac{d^\alpha (e_\alpha^\rho)}{dE_\alpha^\alpha} = \frac{\rho^1}{(r-\alpha)!} E^{\rho-\alpha}, \]  

(6.11)

\[ \frac{(dE_\alpha)^\alpha}{dE_\alpha^\alpha} = (1-\alpha)! E^{\alpha-1}, \]  

(6.12)

and

\[ \frac{d^\alpha E_\alpha}{dh^\alpha} = E_\alpha \left( \frac{\lambda \frac{h^\alpha}{\rho^\alpha}}{\rho^\alpha} \right)^\frac{\lambda}{\rho^\alpha}, \]  

(6.13)

therefore

\[ \frac{d^\alpha}{dh^\alpha} \left[ E_\alpha \left( \lambda \frac{h^\alpha}{\rho^\alpha} \right)^\rho \right] = \frac{\rho^1}{(r-\alpha)!} E_\alpha^{\rho-\alpha} (1-\alpha)! E_\alpha^{\alpha-1} E_\alpha^{\rho} \left( \lambda \frac{h^\alpha}{\rho^\alpha} \right)^\frac{\lambda}{\rho^\alpha} \]  

(6.14)
In other words, we come across the equation \( E \), but slightly modified in the form

\[
y^{(\alpha)}(x) = \lambda \frac{\rho!(1-\alpha)!}{\rho^\alpha(\rho-\alpha)!} y(x).
\]

The fact that we do not recover exactly the same differential equation with the same parameter \( \lambda \) should not be too much surprising and is a result of the approximation in (6.3).

7. **An approximation to the solution of the equation \( E \)**

7.1 **Derivation**

A numerical approximation of the solution of the equation \( E \) can be obtained by combining the fractional Taylor’s series (with Landau’s notation)

\[
y(x + h) = y(x) + \frac{1}{\alpha!} y^{(\alpha)}(x) h^\alpha + o(h^\alpha), \quad (7.1)
\]

with the equation \( E \) that we recall for convenience

\[
y^{(\alpha)}(x) = \lambda y(x). \quad (7.2)
\]

We seek a solution on the lattice defined by the span \( h, i.e: 0, h, 2h, 3h, ... \)

(Step 1) On taking account of (7.2), (7.1) can be re-written in the form

\[
y(x + h) \cong y(x) \left(1 + \frac{\lambda}{\alpha!} h^\alpha \right).
\]

(Step 2) Again one has

\[
y(x + 2h) \cong y(x + h) \left(1 + \frac{\lambda}{\alpha!} h^\alpha \right)
\cong y(x) \left(1 + \frac{\lambda}{\alpha!} h^\alpha \right)^2. \quad (7.3)
\]

(Step 3) Recursively one obtains

\[
y(x + kh) \cong y(x) \left(1 + \frac{\lambda}{\alpha!} h^\alpha \right)^k \quad (7.4)
\]

and

\[
y(x + kh + H) \cong y(x) \left(1 + \frac{\lambda}{\alpha!} h^\alpha \right)^k \left(1 + \frac{\lambda}{\alpha!} H^\alpha \right).
\]

(Step 4) As we are looking for the derivative of \( y(x + kh) \), we need to calculate the difference

\[
\Delta y(x + kh) = y(x + kh + H) - y(x + kh)
\]

\[
= y(x) \left(1 + \frac{\lambda}{\alpha!} H^\alpha \right) \left(1 + \frac{\lambda}{\alpha!} h^\alpha \right)^k - y(x) \left(1 + \frac{\lambda}{\alpha!} h^\alpha \right)^k
\]
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\[ y(x) = y(x) \left(1 + \frac{\lambda}{a!} h^a\right)^k \frac{H^a}{a!} \]
	herefore

\[ \frac{\Delta y(x+kh)}{H} = y(x) \left(1 + \frac{\lambda}{a!} h^a\right)^k \frac{H^a-1}{a!} \]

with \(0 < \alpha < 1\).

We conclude that \(y(x + kh)\) is not differentiable at \(x + kh\).

7.2 Relation with log-self-similarity

A real process \(y(x)\) is referred to as a self-similar process of index \(H\) (do not confuse with the span above!) if for each positive \(a\) the distributions of \(\{y(ax)\}\) and \(\{a^H y(x)\}\) are the same. Formally

\[ y(ax) = a^H y(x). \]

and one has \(y(0) = 0\). The parameter \(H\) so introduced is usually referred to as the Hurst’s parameter of the process, and it is closely related to its fractal nature. This being the case, on setting \(a = 1/\rho\), we can re-write the equation (6.8) in the form

\[ y(ax) = (y(x))^a \]

or again

\[ \ln(y(ax)) = a \ln(y(x)) \]

and we arrive at the conclusion that the logarithm of the process is self-similar with a Hurst exponent equal to the unity.

8. Decomposition of the solution of the Equation E

For convenience, let us denote by \(y(t, \lambda)\) the solution of the equation \(E\). This being the case, let us look for a solution in the form

\[ y(t, \lambda) = y_1(x, \lambda_1)y_2(x, \lambda_2), \quad (8.1) \]

where \(y, y_1\) and \(y_2\) are non-differential functions and \(\lambda_1, \lambda_2\) denote parameters to be determined.

(Step 1) Substituting (8.1) into (1.1) yields (via the fractional Leibniz rule which holds for non-differentiable functions)

\[ y_1^{(\alpha)}y_2 + y_1y_2^{(\alpha)} = \lambda y_1y_2 \quad (8.2) \]

therefore

\[ \frac{y_1^{(\alpha)}}{y_1} + \frac{y_2^{(\alpha)}}{y_2} = \lambda. \quad (8.3) \]
(Step 2) We are then led to select $\lambda_1$ and $\lambda_2$ in such a manner to have

$$\lambda_1 + \lambda_2 = \lambda,$$

and to define $y_1$ and $y_2$ by the equations

$$y_1^{(a)}(x, \lambda_1) = \lambda_1 y_1(x, \lambda_1), \quad (8.4)$$
$$y_2^{(a)}(x, \lambda_2) = \lambda_2 y_2(x, \lambda_2). \quad (8.5)$$

(Step 3) In the special case when $\lambda_1 = \lambda_2 = \lambda/2$ we find that

$$y(x, \lambda) = y^2 \left( x, \frac{\lambda}{2} \right). \quad (8.6)$$

(Step 4) More generally, with a decomposition of order $n$, we would have the equality

$$y(x, \lambda) = y^n \left( x, \frac{\lambda}{n} \right). \quad (8.7)$$

which exhibits some similarities with the functional equation (6.7).

9. Concluding remarks

General remarks. In this short paper, we have displayed some remarks and comments related to a basic fractional differential equation, the equation $E$, which is useful in fractional calculus. The stake at issue is as to whether this equation can be considered in the space of non-differentiable functions, and if it can, to outline the kind of results one may so expect to obtain. There are many traps, there are many inconsistencies in fractional calculus, like the non-commutativity of fractional derivatives for instance, which make questionable such questions like the uniqueness of solutions, and we have to cope with this difficulty. We postulated the existence of a solution which is continuous everywhere but is nowhere differentiable, and of course there remains to arrive at the modelling of such a function. For instance one could start from the equation (4.5) and use the approximation

$$y_{n+1} = y_n E_a(h^2).$$

Approach via Green function. Another proposal is to assume that, locally near any value $x_0$, $y(x)$ can be approximated by the function $K(x - x_0)^a$, therefore we may look for a solution in the form (or something like)

$$y(x) = \int_a^b G(\xi)(x - \xi)^a d\xi.$$
Could we use a randomization technique? Given that the study involves functions which are nowhere differentiable, one may ask whether the Gaussian white noise $w(t)$ would not be of some help in this study, in which case we would then have to consider integrals like, for instance,

$$D^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\tau)^{-\alpha} w(\tau) d\tau.$$ 

The reference below should be rather considered as a mini bibliography on the theory and applications of fractional calculus, given that the main textbook to which the present short paper refers to is the 20th reference.

References


On the non-differentiability of the solutions...


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