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Disconnection Probability of Graph on Two Dimensional Manifold: Continuation

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Abstract

In this paper a technique of disconnection probability asymptotic analysis for a random graph placed on two dimensional manifold is developed in details. This technique is based on a lower bound for lengths of graph belts which are uncontractible to a point on the manifold.

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Introduction

In [1] a qualified asymptotic formula for a disconnection probability of a graph with high reliable edges on a two dimensional manifold was obtained.

Assume that G is an undirected connected graph on the two dimensional manifold S (so that its edges may intersect only in final points). A cut of the graph G is a set of edges which deletion makes this graph disconnected. A cocycle is a cut minimal by theoretically set inclusion in the graph G. Denote m the minimal cocycle length in the graph G and put

$$l = l(m) = \min\left\{k : \ k = 1, 2, \dots, \ k \geq \frac{3m}{2}\right\}.$$

Duality condition. For any $k, \ m \leq k < l(m)$, there is one-to-one correspondence between cocycles with a length k in the graph G and simple cycles in the dual graph G^*.

Theorem 0.1 *Assume that edges of the graph G fail with the probability $h \to 0$ and the duality condition is true. Then for the disconnection probability P_h of the graph G the following asymptotic formula takes place:*

$$P_h = \sum_{m \leq k < l} c_k h^k + O(h^l), \ h \to 0. \tag{1}$$

Here c_k is the number of cocycles with the length k.

But Formula (1) is true when the duality condition is true. Earlier a validity check of the duality condition was made heuristically and only in some partial case. In this paper an algorithm of this condition validity check is made for graphs on manifolds of sufficiently general kind.

1 Manifolds and their Euler characteristics

A two dimensional manifold [2] is a two dimensional surface in a three dimensional space where each point may have a vicinity homomorphic with a plane or with a half plane. Manifolds are oriented or disoriented, closed or with a boundary.

In this paper we consider the following two dimensional manifolds [2] - [5]: the connected sum $\mathcal{A}_{n,k}$ of n toruses with k holes, the connected sum $\mathcal{B}_{n,k}$ of

n projective planes with k holes, $n \geq 1,\ k \geq 0$, and the Mobius strip $\mathcal{C}$. The connected sum of two manifolds is created by an excision of small circles from two manifolds and a gluing of them along boundaries of these circles.

Manifolds $\mathcal{A}_{n,0},\ \mathcal{B}_{n,0}$ are closed and manifolds $\mathcal{C},\ \mathcal{A}_{n,k},\ \mathcal{B}_{n,k},\ k > 0$ are manifolds with boundaries. Manifolds $\mathcal{A}_{n,k}$ are oriented and manifolds $\mathcal{C},\ \mathcal{B}_{n,k}$ are disoriented. Remark that manifold $\mathcal{B}_{2,0}$ is the Klein bottle.

Geometrically manifolds $\mathcal{C},\ \mathcal{A}_{n,k},\ \mathcal{B}_{n,k}$ are represented as right quadrilateral, right $(4n+3k)$ - lateral and right $(2n+3k)$ - lateral, relatively, and are described by the following words:

$$\mathcal{C} = cada,\ \mathcal{A}_{n,k} = a_1 b_1 a_1^{-1} b_1^{-1} \dots a_n b_n a_n^{-1} b_n^{-1} c_1 D_1 c_1^{-1} \dots c_k D_k c_k^{-1},$$

$$\mathcal{B}_{n,k} = a_1 a_1 \dots a_n a_n c_1 D_1 c_1^{-1} \dots c_k D_k c_k^{-1}.$$

Here each word contains no more than two identical letters. Each letter characterizes a side of right polygon which are successive anticlockwise. Each side of polygon has its direction: clockwise with power (-1) and anticlockwise with power $(+1)$. Sides of the polygon with identical letters are glued so that their directions are coherent. Here letters $D_i,\ 1 \leq i \leq k$, corresponded boundaries of holes. Listed right polygons are called plane scans of corresponding manifolds [2, Chapter I, Figure 1.28].

Then the graph G on the manifold S may be considered as a set of faces (polygons) which edges may intersect only in final points on the plane scan of this manifold with a gluing of appropriate manifold sides. Edges of faces are glued completely or have not common internal points.

It is known [2, Chapter I, § 8, 11] that for any graph G, on the two dimensional manifold S its Euler characteristic $\chi(S) = v - r + g$, where $v$is a number of nodes, r is a number of edges, g is a number of faces of this graph, are defined only by the manifold S, but not by the graph G. The following equalities are true: $\chi(\mathcal{A}_{n,k}) = 2 - 2n - k,\ \chi(\mathcal{B}_{n,k}) = 2 - n - k,\ n \geq 1,\ k \geq 0,\ \chi(\mathcal{C}) = 0$.

Theorem 1.1 *Minimal power of a node in the graph G on the manifold $\mathcal{S}$, does not exceed* $M(S) = \max\left(k:\ k = 1, 2, \dots,\ k \leq 6 - \dfrac{6\chi(S)}{v}\right)$.

Proof. As each face $A_k,\ 1 \leq k \leq m$, of the graph G contains more than two sides so following [6, Corollary of Theorem 1.6], we obtain the inequality $3g \leq 2r$. Connecting this inequality with the formula for $\chi(S)$, we obtain the inequality $0 \leq 6v - 2r - 6\chi(S)$. Assume that any node of the graph G has a power larger or equal k, then $kv - 2r \leq 0$. Consequently maximal possible quantity k is defined by the relation $k = M(S)$.

Remark 1.2 *The minimal cocycle length m in the graph G does not exceed $M(S)$. If we know $M(S)$, then using the inequality $m \leq M(s)$ and Formulas [7, 8] for a calculation of constants c_k, $m \leq k < l(m)$, it is possible to estimate a number of arithmetic operations for a calculation of these constants.*

2 Validity check of duality condition

Separate out in dual graph G^* simple cycle $\Gamma^* = (A_1, A_2, \dots, A_k, A_1)$, consisting of faces $A_1, \dots, A_k$ - nodes of this cycle and edges $r_1 = (A_1, A_2), \dots, r_{k-1} = (A_{k-1}, A_k)$, $r_k = (A_k, A_1)$, $k < l$, connecting neighbor nodes of the cycle Γ^*. Fix a mid of the edge r_1 and connect it with a mid of the edge r_2 by the curve Γ_1, (nonintersected, continuous and pise smooth) in an interior of the face A_2. Analogously define the curves $\Gamma_2, \dots, \Gamma_k$, connecting middles of edges r_2, $r_3, \dots, r_k, r_1$, correspondingly. The curves $\Gamma_1, \dots, \Gamma_k$, create the closed curve Γ, in the belt $(A_1, \dots, A_k)$ interior on the manifold S.

Theorem 2.1 *Assume that the closed curve* Γ, *according to the simple cycle Γ^* with the length k, $m \leq k < l$, may be contracted continuously into a point on the manifold S. Then the following statement is true. 1) Each simple cycle Γ^* with the length k, $m \leq k < l$, in the dual graph G^* accords the cocycle R with the length k in the graph G.*
2) Each cocycle R with the length k, $m \leq k < l$, accords simple cycle Γ^* with the length k in the graph G^*.

Proof. 1) Assume that $r_1, \dots, r_k$ is a simple cycle in the dual graph G^*, according the belt $(A_1, \dots, A_k)$ in the graph G. As S is two dimensional manifold then the closed curve Γ according this belt is projected homomorphic on some plane. By Jordan theorem (see for example [9]) this closed curve becomes external to some set U of the graph G nodes. And all nodes of the graph G, which do not belong to the set U, may be connected with nodes from the set U only by ways which contain some edge from the set $\{r_1, \dots, r_k\}$. Consequently the set $\{r_1, \dots, r_k\}$ is a cut in the graph G.

Assume that the cut $\{r_1, \dots, r_k\}$ is not a cocycle, for example after a deletion of its edge r_1 the set $\{r_2, \dots, r_k\}$ becomes a cut also. As this edges set accords the simple way $(A_2, \dots, A_k)$, in the graph G^* then the edge r_k may be bypassed along the boundary of the face A_1 by a way which contains the deleted edge r_1. So the set $r_2, \dots, r_k$ is not a cut. Consequently the cut $\{r_1, \dots, r_k\}$ is a cocycle.

2) Assume that R is a cocycle with the length k, $m \leq k < l$, in the graph G, which containes the edge r. The edge r divides the faces A, B of the graph G on the manifold S. Then the face A contains at least one more edge from the

set R. In opposite case the edge r may be bypassed along a boundary of the face A and so its ends may be connected by a way consisting of edges which do not belong to the set R.

Consider the subgraph G_R^* of the graph G^* with the edges set $R = \{r_1, \ldots, r_k\}$. It is clear that in this subgraph G_R^* each node is connected at least with two edges and so there is a simple way for example $\Gamma^* = (A_1, A_2, \ldots, A_s)$, which consists of the faces $A_1, \ldots, A_s$ - cycle nodes and edges $r_1 = (A_1, A_2), \ldots, r_{s-1} = (A_{s-1}, A_s),\ r_s = (A_s, A_1),\ s < l$. Contrast the simple cycle Γ^* a belt on the manifold S and the curve Γ according this belt. Using considerations of the previous point and Theorem 2.1 conditions we obtain that the set $\{r_1, \ldots, r_s\}$ is a cut and contains in the set R. As R is a cocycle then $\{r_1, \ldots, r_s\} = R,\ s = k$. Consequently the subgraph G_R^* is a simple cycle. Theorem 2.1 is proved.

Corollary 2.2 *If each belt non constricting into a point on the manifold S has a length larger or equal $l(m)$, then Theorem 0.1 is true.*

A check out of the duality condition is based on a representation of all closed curves Γ^*, which can not be contracted into a point on the manifold S, as generators of the fundamental group (Poincare group) [2, Chapter II], [3, Part II, Chapter 4, §17, 18], [4, Chapter III, § 4], [5, Chapter 26, pp. 234] and all curves homotopic them on the manifold S.

Generators of the fundamental group of the manifold $\mathcal{A}_{n,k}$ are curves connecting glued points: of sides $a_i,\ a_i^{-1}$, of sides $b_i,\ b_i^{-1},\ 1 \le i \le n$, of sides $c_j,\ c_j^{-1},\ 0 \le j \le k$, in right polygon representing this manifold. These generators satisfy the connection equality

$$a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_n b_n a_n^{-1} b_n^{-1} c_1 D_1 c_1^{-1} \ldots c_k D_k c_k^{-1} = e,$$

where e is a unit element (a curve contracted into a point).

Generators of the manifold $\mathcal{B}_{n,k}$ are curves connecting glued points: of sides $a_i,\ a_i,\ 1 \le i \le n$, and of sides $c_j,\ c_j^{-1},\ 1 \le j \le k$, in right polygon representing this manifold. These generators satisfy connection equality

$$a_1 a_1 \ldots a_n a_n c_1 D_1 c_1^{-1} \ldots c_k D_k c_k^{-1} = e.$$

3 Minimal length of uncontractible belt

Any closed belt $(A_1, \ldots, A_k)$ (without self intersections and uncontractible on the manifold S to a point) may begin from a face which has an edge on a side a of the manifold S plane scan and ends on a face a' which glues with the face a. Contrast this belt the closed curve Γ, which is a generator of the manifold S fundamental group.

Denote S_a the set which is constructed from the manifold S by a deletion of the sides $a,\ a'$ gluing. Using simple geometric consideration it is possible to establish that the curve Γ has a peace connecting equivalent (glued) points of the sides $a,\ a'$ in the set S_a. Consequently in the belt $A_1, \dots, A_k$ it is possible to separate subsequence of faces $A_t, A_{t+1}, \dots, A_r,\ 1 \le t < r \le k$, completely containing in the set S_a. Here the face A_t has common edge with the side a, and the face A_r has common edge with- the side a' (see Figure 1).

Denote $\mathcal{R}_a$ the set of all sequences of incident faces $A_1, \dots, A_j$ completely containing in the set S_a so that the face A_1 has common edge with the side a, and the face A_j has common edge with equivalent side a'. Assume that $\mathcal{A}(S)$ is the set of all glued sides of the plane scan of the manifold S, put

$$\mathcal{R} = \bigcup_{a \in \mathcal{A}(S)} \mathcal{R}_a,\ s = \min\{k :\ (A_1, \dots, A_k) \in \mathcal{R}\}.$$

Then Corollary 2.2 is true if the inequality $s \ge l(m)$.takes place.

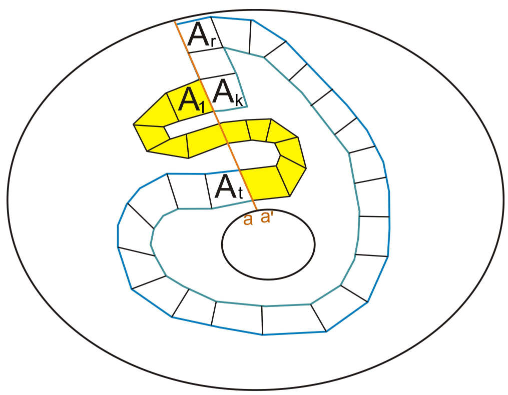

Figure 1. Belt on torus.

The check out of last inequality is based on the following procedure. Consider the graph G_a, which is obtained from the graph G by a deletion of the sides $a,\ a'$ gluing. Contrast this graph its dual graph G_a^*. Separate in the graph G_a^* the sets of nodes $V_a^+,\ V_a^-$, The set V_a^+ consists of the graph G faces which have common edges with the side a. Analogously the set V_a^- is defined. Denote s_a minimal number of nodes in ways connecting in the graph G_a^* nodes from the sets $V_a^+,\ V_a^-$, then $s = \min_{a \in \mathcal{A}(S)} s_a$.

4 Numerical experiment for graphs on manifolds

To calculate the asymptotic formula in an accordance with Theorem 2.1 it is necessary to calculate by the adjacency matrix A of the dual graph G^*, constructed with an accounting of a gluing the minimal length of simple cycles m, the number $l = l(m)$, and the numbers c_k [7, 8] of simple cycles with the length k, $m \le k < l(m)$.

To convince that Theorem 2.1 conditions are true it is enough to calculate s. If $s \geq l(m)$ then the statement of Theorem 0.1 takes place and so it is possible to use the asymptotic formula for the disconnection probability P_h calculation.

Consider examples of connected graphs (see Figures 2, 3) on torus, Mobius strip, projective plane and Klein buttle. In these graphs we assume that edge failure probability $h = 0.02$ and integer constants m, s, l, $c_m, \dots, c_{l-1}$, disconnection probability by asymptotic formula P_h and by Monte-Carlo method P_{MC} with realizations number 10^6, relative error $\delta_h = |P_h - P_{MC}|/P_{MC}$ are calculated. Calculations results are represented in Table 1.

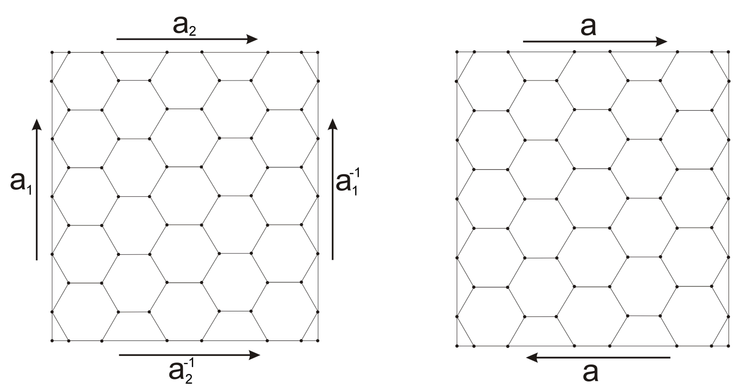

Figure 2. Graph on torus (on the left), on Mobius strip (on the right).

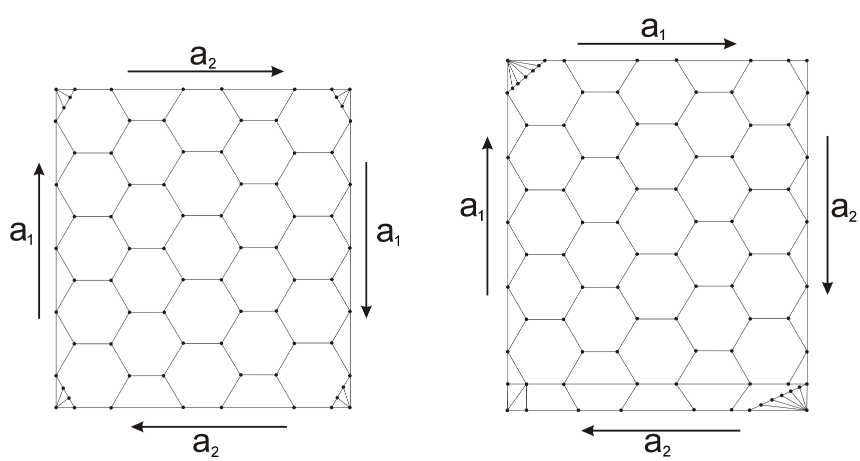

Figure 3. Graph on projective plane (on the left), on Klein buttle (on the right).

Manifold	m	s	l	c_3	c_4	P_h	P_{MC}	Δ_h
Torus	3	5	5	42	57	0.00034512	0.000361	0.043
Mobius strip	3	5	5	43	57	0.000353	0.000356	0.00809
Projective plane	3	5	5	51	63	0.000418	0.000410	0.019
KLein buttle	3	6	5	59	69	0.000483	0.000472	0.023

Table 1. Results of calculations.

Constants c_3, c_4 are calculated by the formulas
$c_3 = \frac{1}{6}trA^3$, $c_4 = \frac{1}{8}\left(trA^4 - 2m - 2\sum_{1\leq i\neq j\leq n} a_{ij}^{(2)}\right)$, where A is the contiguity matrix of the dual graph G^*, $a_{ij}^{(2)}$ is an element of the matrix A^2.

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