Continuous Dependence of the Solution of
Itô Stochastic Differential Equation with
Nonlocal Conditions

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Abstract

In this paper we are concerned with a nonlocal problem of a stochas-
tic differential equation of Itô type. The solution contains both of Rie-
mann (or Lebesgue) and Itô integrals in the mean square sense, so we
study the existence of a unique mean square continuous solution and its
continuous dependence on the random data $X_0$ and on the (non-random
data) coefficients of the nonlocal condition $a_k$. Also, a stochastic differ-
ential equation with the integral condition will be considered.

Keywords: Riemann integral, Itô integral, Brownian motion, nonlocal
condition, unique mean square solution, continuous dependence, random data,
non-random data, integral condition

1 Introduction

Stochastic differential equations have been extensively studied by several authors
specially stochastic differential equations of Itô’s type, they studied Itô’s integral
in mean square sense as in ([19]) and in almost certain sense as in ([2]), properties
of Brownian motion ( or a Wiener process) as a formal derivative of the Gaussian
white noise occupied much attention of authors. A Brownian motion $W(t), t \in \mathbb{R}$, is defined as a stochastic process such that

$$W(0) = 0, \quad E(W(t)) = 0, \quad E(W(t))^2 = t$$

and $[W(t_1) - W(t_2)]$ is a Gaussian random variable for all $t_1, t_2 \in \mathbb{R}$.

The reader is referred to ([1]-[3]), ([7]) and ([12]-[21]) and references therein. Also problems with nonlocal conditions have been heavily studied by several authors in the last decades in the ordinary differential equations. The reader is referred to ([4]-[6]) and ([8]-[11]), and references therein.

Here we are concerned with the stochastic differential equation of Itô’s type

$$dX(t) = f(t, X(t))dt + g(t, X(t))dW(t), \quad t \in (0, T] \tag{1}$$

with the nonlocal random initial condition

$$X(0) + \sum_{k=1}^{n} a_k X(\tau_k) = X_0, \quad a_k > 0, \quad \tau_k \in (0, T), \tag{2}$$

where $X_0$ is a second order random variable independent of the Brownian motion (or Wiener process) $W(t)$ and $a_k$ are positive real numbers.

The existence of a unique mean square continuous solution will be studied. The continuous dependence on the random data $X_0$ and the non-random data $a_k$ will be established. The problem (1) with the mean square Riemann-Steltjes integral condition

$$X(0) + \int_{0}^{T} X(s)dv(s) = X_0. \tag{3}$$

will be considered.

2 **Existence and uniqueness**

Let $I = [0, T]$ and $C = C(I, L_2(\Omega))$ be the class of all mean square continuous second order stochastic process with the norm

$$\|X\|_C = \sup_{t \in [0,T]} \|X(t)\|_2 = \sup_{t \in [0,T]} \sqrt{E(X(t))^2}.$$

Throughout the paper we assume that the following assumptions hold

(H1) The functions $f : [0, T] \times L_2(\Omega) \to L_2(\Omega)$ and $g : [0, T] \times L_2(\Omega) \to L_2(\Omega)$ are mean square continuous, and there exists a positive real numbers $r_1$ and $r_2$ such that

$$\sup_{t \in [0,T]} |f(t,0)| \leq r_1, \quad \sup_{t \in [0,T]} |g(t,0)| \leq r_2.$$
(H2) There exists an integrable functions $k_1 : [0, T] \to R^+$ and $k_2 : [0, T] \to R^+$, where

$$\sup_{t \in [0, T]} \int_0^t k_1(s)ds \leq m_1, \quad \sup_{t \in [0, T]} \int_0^t k_2(s)ds \leq m_2,$$

such that the function $f$ and $g$ satisfy the mean square Lipschitz condition

$$\| f(t, X_1(t)) - f(t, X_2(t)) \|_2 \leq k_1(t) \| X_1(t) - X_2(t) \|_2$$

and

$$\| g(t, X_1(t)) - g(t, X_2(t)) \|_2 \leq k_2(t) \| X_1(t) - X_2(t) \|_2.$$

Now we have the following lemma.

**Lemma 2.1** The solution of the nonlocal stochastic problem (1)-(2) can be expressed by the stochastic integral equation

$$X(t) = a \left( X_0 - \sum_{k=1}^n a_k \int_0^\tau_k f(s, X(s))ds + \sum_{k=1}^n a_k \int_0^\tau_k g(s, X(s))dW(s) \right) + \int_0^t f(s, X(s))ds + \int_0^t g(s, X(s))dW(s),$$

(4)

where $a = \left( 1 + \sum_{k=1}^n a_k \right)^{-1}$.

**Proof.** Integrating equation (1), we obtain

$$X(t) = X(0) + \int_0^t f(s, X(s))ds + \int_0^t g(s, X(s))dW(s),$$

then

$$\sum_{k=1}^n a_k X(\tau_k) = \sum_{k=1}^n a_k X(0) + \sum_{k=1}^n a_k \int_0^{\tau_k} f(s, X(s))ds + \sum_{k=1}^n a_k \int_0^{\tau_k} g(s, X(s))dW(s)$$

and

$$X_0 - X(0) = \sum_{k=1}^n a_k X(0) + \sum_{k=1}^n a_k \int_0^{\tau_k} f(s, X(s))ds + \sum_{k=1}^n a_k \int_0^{\tau_k} g(s, X(s))dW(s)$$

and

$$\left( 1 + \sum_{k=1}^n a_k \right) X(0) = X_0 - \sum_{k=1}^n a_k \int_0^{\tau_k} f(s, X(s))ds - \sum_{k=1}^n a_k \int_0^{\tau_k} g(s, X(s))dW(s),$$
then

\[ X(0) = \left(1 + \sum_{k=1}^{n} a_k\right)^{-1} \left( X_0 - \sum_{k=1}^{n} a_k \int_{0}^{\tau_k} f(s, X(s)) ds - \sum_{k=1}^{n} a_k \int_{0}^{\tau_k} g(s, X(s)) dW(s) \right). \]

Hence

\[ X(t) = a \left( X_0 - \sum_{k=1}^{n} a_k \int_{0}^{\tau_k} f(s, X(s)) ds - \sum_{k=1}^{n} a_k \int_{0}^{\tau_k} g(s, X(s)) dW(s) \right) \]

\[ + \int_{0}^{t} f(s, X(s)) ds + \int_{0}^{t} g(s, X(s)) dW(s), \]

where \( a = \left(1 + \sum_{k=1}^{n} a_k\right)^{-1}. \]

Now define the mapping

\[ AX(t) = a \left( X_0 - \sum_{k=1}^{n} a_k \int_{0}^{\tau_k} f(s, X(s)) ds - \sum_{k=1}^{n} a_k \int_{0}^{\tau_k} g(s, X(s)) dW(s) \right) \]

\[ + \int_{0}^{t} f(s, X(s)) ds + \int_{0}^{t} g(s, X(s)) dW(s). \]

Then we can prove the following lemma.

**Lemma 2.2** \( A : C \to C \).

**Proof.** Let \( X \in C, \ t_1, t_2 \in [0, T] \) such that \( |t_2 - t_1| < \delta \), then

\[ AX(t_2) - AX(t_1) = \int_{t_1}^{t_2} f(s, X(s)) ds + \int_{t_1}^{t_2} g(s, X(s)) dW(s). \]

From assumption (H2) we have

\[ \| f(t, X(t)) \|_2 - |f(t, 0)| \leq \| f(t, X(t)) - f(t, 0) \|_2 \leq k_1(t) \| X(t) \|_2, \]

then we have

\[ \| f(t, X(t)) \|_2 \leq k_1(t) \| X \|_C + r_1 \]

and similarly,

\[ \| g(t, X(t)) \|_2 \leq k_2(t) \| X \|_C + r_2, \]

then we have

\[ \| AX(t_2) - AX(t_1) \|_2 \leq \int_{t_1}^{t_2} \| f(s, X(s)) ds \|_2 + \int_{t_1}^{t_2} \| g(s, X(s)) dW(s) \|_2. \]
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Now
\[
\left\| \int_{t_1}^{t_2} f(s, X(s)) ds \right\|_2 \leq \int_{t_1}^{t_2} \| f(s, X(s)) \|_2 ds \leq \int_{t_1}^{t_2} [k_1(s) \| X \|_C + r_1] ds
\]
\[
\leq \| X \|_C \int_{t_1}^{t_2} k_1(s) ds + r_1(t_2 - t_1)
\]
and
\[
\left\| \int_{t_1}^{t_2} g(s, X(s)) dW(s) \right\|_2 \leq \int_{t_1}^{t_2} \| g(s, X(s)) \|_2 \| X \|_C + g(s, 0) \|_2 ds
\]
\[
\leq 2 \| X \|_C \int_{t_1}^{t_2} k_2(s) ds + 2 \int_{t_1}^{t_2} | g(s, 0) |^2 ds
\]
\[
\leq 2 \| X \|_C \int_{t_1}^{t_2} k_2(s) ds + 2r_2^2(t_2 - t_1).
\]

So,
\[
\| A X(t_2) - A X(t_1) \|_2 \leq \| X \|_C \int_{t_1}^{t_2} k_1(s) ds + r_1(t_2 - t_1) + \| X \|_C \sqrt{2 \int_{t_1}^{t_2} k_2^2(s) ds + r_2 \sqrt{2(t_2 - t_1)}},
\]
which proves that $A : C \to C$.  

For the existence of a unique mean square continuous solution $X \in C$ of the problem (1)-(2), we have the following theorem.

**Theorem 2.1** Let the assumptions (H1)-(H2) be satisfied. If $2(m_1 + m_2) < 1$, then the problem (1)-(2) has a unique solution $X \in C$.

**Proof**. Let $X$ and $X^* \in C$, then
\[
\| AX(t) - AX^*(t) \|_2 \leq \| \int_0^t [f(s, X(s)) - f(s, X^*(s))] ds - a \sum_{k=1}^n a_k \int_0^{\tau_k} [f(s, X(s)) - f(s, X^*(s))] ds \|_2
\]
\[
+ \| \int_0^t [g(s, X(s)) - g(s, X^*(s))] dW(s) - a \sum_{k=1}^n a_k \int_0^{\tau_k} [g(s, X(s)) - g(s, X^*(s))] dW(s) \|_2.
\]

we have
\[
\| \int_0^t [f(s, X(s)) - f(s, X^*(s))] ds \|_2 \leq \| X - X^* \|_C \int_0^t k_1(s) ds \leq m_1 \| X - X^* \|_C
\]
and
\[
\left\| \int_0^t (g(s, X(s)) - g(s, X^*(s))) dW(s) \right\|_2^2 = \int_0^t \left\| g(s, X(s)) - g(s, X^*(s)) \right\|_2^2 ds \\
\leq \| X - X^* \|_C^2 \int_0^t |k_2(s)|^2 ds \leq m_2^2 \| X - X^* \|_C^2.
\]

Hence
\[
\| AX - AX^* \|_C \leq 2(m_1 + m_2) \| X - X^* \|_C.
\]

If \(2(m_1 + m_2) < 1\), then \(A\) is contraction and there exists a unique solution \(X \in C\) of the nonlocal stochastic problem (1)-(2), [3]. This solution is given by (4). 

3 Continuous dependence

Consider the stochastic differential equation (1) with the nonlocal condition
\[
X(0) + \sum_{k=1}^n a_k X(\tau_k) = \tilde{X}_0, \quad \tau_k \in (0, T) \tag{5}
\]

**Definition 3.1** The solution \(X \in C\) of the nonlocal problem (1), (2) is continuously dependent (on the data \(X_0\)) if \(\forall \epsilon > 0, \exists \delta > 0\) such that \(\| X_0 - \tilde{X}_0 \|_2 \leq \delta\) implies that \(\| X - \tilde{X} \|_C \leq \epsilon\)

Here, we study the continuous dependence (on the random data \(X_0\)) of the solution of the stochastic differential equation (1) and (2).

**Theorem 3.2** Let the assumptions (H1)-(H2) be satisfied. Then the solution of the nonlocal problem (1)-(2) is continuously dependent on the random data \(X_0\).

**Proof.** Let \(X(t)\) as defined in equation (4) be the solution of the nonlocal problem (1)-(2) and
\[
\tilde{X}(t) = a \left( \tilde{X}_0 - \sum_{k=1}^n a_k \int_0^{\tau_k} f(s, \tilde{X}(s)) ds - \sum_{k=1}^n a_k \int_0^{\tau_k} g(s, \tilde{X}(s)) dW(s) \right) \\
+ \int_0^t f(s, \tilde{X}(s)) ds + \int_0^t g(s, \tilde{X}(s)) dW(s)
\]
be the solution of the nonlocal problem (1) and (5). Then
\[
X(t) - \tilde{X}(t) =
\]
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\[ a[X_0 - \tilde{X}_0] - a \sum_{k=1}^{n} a_k \int_{0}^{\tau_k} [f(s, X(s)) - f(s, \tilde{X}(s))] ds \]

\[ -a \sum_{k=1}^{n} a_k \int_{0}^{\tau_k} [g(s, X(s)) - g(s, \tilde{X}(s))] dW(s) \]

\[ + \int_{0}^{t} [f(s, X(s)) - f(s, \tilde{X}(s))] ds + \int_{0}^{t} [g(s, X(s)) - g(s, \tilde{X}(s))] dW(s). \]

Using our assumptions, we get

\[ \| X(t) - \tilde{X}(t) \|_2 \leq a \| X_0 - \tilde{X}_0 \|_2 + a \sum_{k=1}^{n} a_k \int_{0}^{\tau_k} \| f(s, X(s)) - f(s, \tilde{X}(s)) \|_2 ds \]

\[ + a \sum_{k=1}^{n} a_k \left\| \int_{0}^{\tau_k} [g(s, X(s)) - g(s, \tilde{X}(s))] dW(s) \right\|_2 \]

\[ + \int_{0}^{t} \| f(s, X(s)) - f(s, \tilde{X}(s)) \|_2 ds + \left\| \int_{0}^{t} [g(s, X(s)) - g(s, \tilde{X}(s))] dW(s) \right\|_2 \]

then we can get

\[ \| X - \tilde{X} \|_C \leq a\delta + a \sum_{k=1}^{n} a_k m_1 \| X - \tilde{X} \|_C + a \sum_{k=1}^{n} a_k m_2 \| X - \tilde{X} \|_C \]

\[ + m_1 \| X - \tilde{X} \|_C + m_2 \| X - \tilde{X} \|_C \]

\[ \leq a\delta + 2(m_1 + m_2) \| X - \tilde{X} \|_C \]

Hence

\[ \| X - \tilde{X} \|_C \leq \frac{a\delta}{1 - 2(m_1 + m_2)}. \]

This complete the proof. 

Now consider the stochastic differential equation (1) with the nonlocal condition

\[ X(0) + \sum_{k=1}^{n} \tilde{a}_k X(\tau_k) = X_0, \quad \tau_k \in (0, T) \quad (6) \]

**Definition 3.2** The solution \( X \in C \) of the nonlocal problem (1)-(2) is continuously dependent (on the data \( a_k \)) if \( \forall \epsilon > 0 \), \( \exists \delta > 0 \) such that \( | a_k - \tilde{a}_k | \leq \delta \) implies that \( \| X - \tilde{X} \|_C \leq \epsilon \)

Here, we study the continuous dependence (on the coefficient \( a_k \) of the nonlocal condition) of the solution of the stochastic differential equation (1) and (2).
Theorem 3.3 Let the assumptions (H1)-(H2) be satisfied. Then the solution of the nonlocal problem (1)-(2) is continuously dependent on the coefficients \( a_k \) of the nonlocal condition.

Proof. Let \( X(t) \) as defined in equation (4) be the solution of the nonlocal problem (1)-(2) and

\[
X(t) = \tilde{a} \left( X_0 - \sum_{k=1}^{n} \tilde{a}_k \int_0^{\tau_k} f(s, \tilde{X}(s)) ds - \sum_{k=1}^{n} a_k \int_0^{\tau_k} g(s, X(s)) dW(s) \right) + \int_0^t f(s, \tilde{X}(s)) ds + \int_0^t g(s, \tilde{X}(s)) dW(s)
\]

be the solution of the nonlocal problem (1) and (6). Then

\[
X(t) - \tilde{X}(t) = [a - \tilde{a}]X_0 + \int_0^t [f(s, X(s)) - f(s, \tilde{X}(s))] ds + \int_0^t [g(s, X(s)) - g(s, \tilde{X}(s))] ds \\
- a \sum_{k=1}^{n} a_k \int_0^{\tau_k} f(s, X(s)) ds + \tilde{a} \sum_{k=1}^{n} \tilde{a}_k \int_0^{\tau_k} f(s, \tilde{X}(s)) ds \\
- a \sum_{k=1}^{n} a_k \int_0^{\tau_k} g(s, X(s)) ds + \tilde{a} \sum_{k=1}^{n} \tilde{a}_k \int_0^{\tau_k} g(s, \tilde{X}(s)) ds.
\]

Now

\[
|a - \tilde{a}| = \left| \frac{1}{1 + \sum_{k=1}^{n} a_k} - \frac{1}{1 + \sum_{k=1}^{n} \tilde{a}_k} \right| = \left| \sum_{k=1}^{n} (\tilde{a}_k - a_k) \right| \leq \sum_{k=1}^{n} |\tilde{a}_k - a_k| \leq n\delta,
\]
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and

\[ \tilde{a} \sum_{k=1}^{n} \tilde{a}_k \int_{0}^{\tau_k} f(s, \tilde{X}(s))ds - a \sum_{k=1}^{n} a_k \int_{0}^{\tau_k} f(s, X(s))ds = \tilde{a} \left( 1 + \sum_{k=1}^{n} \tilde{a}_k \right) \int_{0}^{\tau_k} f(s, \tilde{X}(s))ds \]

\[ - a \left( 1 + \sum_{k=1}^{n} a_k \right) \int_{0}^{\tau_k} f(s, X(s))ds - \tilde{a} \int_{0}^{\tau_k} f(s, \tilde{X}(s))ds + a \int_{0}^{\tau_k} f(s, X(s))ds \]

\[ = \tilde{a}(\tilde{a}^{-1}) \int_{0}^{\tau_k} f(s, \tilde{X}(s))ds - a(\tilde{a}^{-1}) \int_{0}^{\tau_k} f(s, X(s))ds - \tilde{a} \int_{0}^{\tau_k} f(s, \tilde{X}(s))ds + a \int_{0}^{\tau_k} f(s, X(s))ds \]

\[ = - \int_{0}^{\tau_k} [f(s, X(s)) - f(s, \tilde{X}(s))]ds + a \int_{0}^{\tau_k} f(s, X(s))ds - \tilde{a} \int_{0}^{\tau_k} f(s, \tilde{X}(s))ds + \tilde{a} \int_{0}^{\tau_k} [f(s, X(s)) - f(s, \tilde{X}(s))]ds, \]

similarly

\[ \tilde{a} \sum_{k=1}^{n} \tilde{a}_k \int_{0}^{\tau_k} g(s, \tilde{X}(s))ds - a \sum_{k=1}^{n} a_k \int_{0}^{\tau_k} g(s, X(s))ds \]

\[ = - \int_{0}^{\tau_k} [g(s, X(s)) - g(s, \tilde{X}(s))]ds + [a - \tilde{a}] \int_{0}^{\tau_k} g(s, X(s))ds + \tilde{a} \int_{0}^{\tau_k} [g(s, X(s)) - g(s, \tilde{X}(s))]ds. \]

Then,

\[ \| X(t) - \tilde{X}(t) \|_2 \leq \]

\[ n\delta \| X_0 \|_2 + \int_{\tau_k}^{t} \| f(s, X(s)) - f(s, \tilde{X}(s)) \|_2 ds + \left\| \int_{\tau_k}^{t} [g(s, X(s)) - g(s, \tilde{X}(s))]ds \right\|_2 \]

\[ + n\delta \int_{0}^{\tau_k} \| f(s, X(s)) \|_2 ds + \tilde{a} \int_{0}^{\tau_k} \| f(s, \tilde{X}(s)) \|_2 ds \]

\[ + n\delta \left\| \int_{0}^{\tau_k} g(s, X(s))ds \right\|_2 + \tilde{a} \left\| \int_{0}^{\tau_k} [g(s, X(s)) - g(s, \tilde{X}(s))]ds \right\|_2. \]

Using our assumptions we get

\[ \| X - \tilde{X} \|_C \leq n\delta \| X_0 \|_2 + m_1 \| X - \tilde{X} \|_C + m_2 \| X - \tilde{X} \|_C + n\delta (\| X \|_C m_1 + r_1 T) \]

\[ + \tilde{a} m_1 \| X - \tilde{X} \|_C + n\delta \sqrt{2(\| X \|_C m_2 + r_2 \sqrt{T})} + \tilde{a} m_2 \| X - \tilde{X} \|_C, \]
and
\[
\| X - \tilde{X} \|_C \leq n\delta \left[ \| X_0 \|_C + m_1 \| X \|_C + r_1 T + \sqrt{2} \left( \| X \|_C m_2 + r_2 \sqrt{T} \right) \right] \\
+ \left[ m_1 + \tilde{a} m_1 + m_2 + \tilde{a} m_2 \right] \| X - \tilde{X} \|_C \\
\leq n\delta \left[ \| X_0 \|_C + m_1 \| X \|_C + r_1 T + \sqrt{2} \left( \| X \|_C m_2 + r_2 \sqrt{T} \right) \right] + 2(m_1 + m_2) \| X - \tilde{X} \|_C.
\]

Hence
\[
\| X - \tilde{X} \|_C \leq \frac{n\delta \left[ \| X_0 \|_C + m_1 \| X \|_C + r_1 T + \sqrt{2} \left( \| X \|_C m_2 + r_2 \sqrt{T} \right) \right]}{1 - 2(m_1 + m_2)}.
\]

This complete the proof. ■

4 Nonlocal integral condition

Let \( v : [0, T] \to [0, T] \) be nondecreasing function such that \( a_k = v(t_k) - v(t_{k-1}), \tau_k \in (t_{k-1}, t_k), \) where \( 0 < t_1 < t_2 < t_3 < \ldots < T \). Then, the nonlocal condition (2) will be in the form
\[
X(0) + \sum_{k=1}^{n} X(\tau_k) (v(t_k) - v(t_{k-1})) = X_0.
\]

From the mean square continuity of the solution of the nonlocal problem (1)-(2), we obtain from [19]
\[
\lim_{n \to \infty} \sum_{k=1}^{n} X(\tau_k) (v(t_k) - v(t_{k-1})) = \int_0^T X(s)dv(s),
\]
that is, the nonlocal conditions (2) is transformed to the mean square Riemann-Steltjes integral condition
\[
X(0) + \int_0^T X(s)dv(s) = X_0,
\]
Now, we have the following theorem.

**Theorem 4.4** Let the assumptions (H1)-(H2) be satisfied, then the stochastic differential equation (1) with the nonlocal integral condition (3) has a unique solution represented in the form
\[
X(t) = a^* \left( X_0 - \int_0^T \int_0^s f(\theta, X(\theta))d\theta dv(s) - \int_0^T \int_0^s g(\theta, X(\theta))dW(\theta)dv(s) \right) \\
+ \int_0^t f(\theta, X(\theta))d\theta + \int_0^t g(\theta, X(\theta))dW(\theta).
\]

where \( a^* = (1 + v(T) - v(0))^{-1}. \)
Proof. Taking the limit of equation (4) we get the proof. ■

References


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