On the Span and Extent of Unit-Distance Graphs in the Plane

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Abstract

Let $G$ be a unit-distance graph in $\mathbb{R}^2$. For each unit-distance representation of $G$ in $\mathbb{R}^2$, there is a smallest circumscribing circle. The infimum of the diameters of these circles, taken over all unit-distance representations of $G$, is called the span of $G$. On the other hand, the supremum of the diameters of all such circles is called the extent of $G$.

We show that the ratio of the extent to the diameter can be made arbitrarily small. Also, we prove that the extent of $G$ does not exceed $\frac{2}{3}\sqrt{3}$ times the graph theoretic diameter of $G$. We further show that for every integer $d \geq 1$, there exists a unit-distance graph $G$ in $\mathbb{R}^2$ with diameter $d$ and extent equal to $\frac{2}{3}\sqrt{3}d$.

Keywords: unit graphs, span of graphs, extent of graphs
1 Introduction and Preliminary Notes

By a graph $G$ we shall mean an ordered pair $G = (V, E)$, where $V = V(G)$ is a finite nonempty set of elements called vertices while $E = E(G)$ is a set of 2-subsets of $V$, called edges. A 2-subset $\{x, y\}$ shall be denoted by the symbol $[x, y]$ or $[y, x]$. We say that $x$ and $y$ are adjacent if $[x, y]$ is an edge.

If $p = (a_1, a_2, ..., a_n)$ and $q = (b_1, b_2, ..., b_n)$ are two points in $\mathbb{R}^n$, we define and denote the Euclidean distance between $p$ and $q$ by $|pq| = \sqrt{\sum_{i=1}^{n} (a_i - b_i)^2}$. We shall simply say distance instead of Euclidean distance. The notation $d(x, y)$ means the graph-theoretic distance between vertices $x$ and $y$.

**Definition 1.1** Let $G$ be a graph. A unit-distance representation of $G$ in the Euclidean space $\mathbb{R}^n$ is a mapping $\phi: V(G) \rightarrow \mathbb{R}^n$ that is injective and such that $|\phi(x)\phi(y)| = 1$ whenever $[x, y] \in E(G)$. The graph $G$ is a unit-distance graph in $\mathbb{R}^n$ if it has a unit-distance representation in $\mathbb{R}^n$.

The euclidean dimension of a graph $G$ is the minimum $n$ such that $G$ can be embedded as a unit-distance graph in $\mathbb{R}^n$. Maehara and Rödl [9] show that the dimension of any graph is bounded by twice its maximum degree. Erdös, Harary and Tutte [2] determine the dimension of some special graphs. Buckley and Harary [1] give the dimension of a wheel and of a complete tripartite while Maehara [8] give the dimension of a complete multipartite graph.

Other works on unit-distance graphs include subdividing a graph to make it a unit-distance graph in the plane [3], identifying some planar unit-distance graphs having planar unit-distance complement [4], and extending a unit-distance graph into a rigid one [7].

In this paper, span and extent of unit-distance graphs are considered. The formal definition of these properties is given next.

**Definition 1.2** Let $G$ be a unit-distance graph in $\mathbb{R}^n$. For each unit-distance representation of $G$ in $\mathbb{R}^n$, there is a smallest circumscribing ball in $\mathbb{R}^n$. The infimum of the diameters of these balls, taken over all unit-distance representations of $G$ in $\mathbb{R}^n$, is called the span of $G$ in $\mathbb{R}^n$. On the other hand, the supremum of the diameters of all such balls is called the extent of $G$ in $\mathbb{R}^n$.

For simplicity, we will just say unit-distance representation and unit-distance graph instead of unit-distance representation in $\mathbb{R}^2$ and unit-distance graph in $\mathbb{R}^2$, respectively. The readers are referred to [5] for the definitions of some graph theoretic concepts and special graphs.

Lemma 1.3 says that for all pairs of vertices in a unit graph, the Euclidean distance is less than or equal to the graph-theoretic distance.

**Lemma 1.3** If $\phi$ is a unit-distance representation of graph $G$ in $\mathbb{R}^n$, then for every two vertices $x$ and $y$ in $G$, $|\phi(x)\phi(y)| \leq d(x, y)$. 
Proof: Let \( x \) and \( y \) be any two vertices in the graph \( G \). If the graph theoretic distance \( d(x, y) \) is infinite, then the conclusion follows. Otherwise, there is a shortest path \( v_1v_2\ldots v_k, \ v_1 = x \) and \( v_k = y \), joining \( x \) and \( y \). Thus, \( d(x, y) = k - 1 \). Now, by the triangle inequality applied to the points \( \phi(v_1), \phi(v_2), \ldots, \phi(v_k) \), we get \( |\phi(x)\phi(y)| \leq |\phi(v_1)\phi(v_2)| + |\phi(v_2)\phi(v_3)| + \cdots + |\phi(v_{k-1})\phi(v_k)| = k - 1 = d(x, y) \). \( \square \)

2 Span of Unit-Distance Graphs in the Plane

The span of a unit graph \( G \) in \( \mathbb{R}^2 \) will be denoted by \( \text{span} \ G \). The cycle of order \( n \) is denoted by \( C_n \). The span of even cycle in \( \mathbb{R}^2 \) is given in Lemma 2.1. The proof is immediate and is omitted.

Lemma 2.1 Let \( C_n \) be the cycle of even order \( n \). Then

\[
\text{span} \ C_n = \begin{cases} \sqrt{2} & \text{if } n = 4, \\ 1 & \text{if } n > 4 \text{ and } n \text{ is even}. \end{cases}
\]

The \( m \times n \) planar grid, denoted by \( P_m \square P_n \), is defined as the Cartesian product of paths \( P_m \) and \( P_n \). Its span in \( \mathbb{R}^2 \) is given in Theorem 2.2.

Theorem 2.2 The span of planar grid \( P_m \square P_n \) is given by

\[
\text{span} \ P_m \square P_n = \begin{cases} \sqrt{2} & \text{if } m > 1 \text{ and } n > 1, \\ 1 & \text{if } m = 1, \ n > 1 \text{ or } n = 1, \ m > 1. \end{cases}
\]

Proof: The last two cases are trivial. We shall prove the case \( m > 1 \) and \( n > 1 \) by induction on \( n \). Let \( n = 2 \). The planar grid \( P_m \square P_2 \) has a span of \( \sqrt{2} \) as illustrated in Figure 1 with \( m = 4 \).

![Figure 1: span \( P_4 \square P_2 = \sqrt{2} \).](image.png)

In a unit-distance representation of \( P_m \square P_n \) in \( \mathbb{R}^2 \) contained in a closed disk of diameter that is arbitrarily close to \( \sqrt{2} \), each copy of \( P_m \) has sides that almost coincide. This means that the sides of \( P_m \) are contained in a closed disk of diameter arbitrarily close to 1.

Assume that \( P_m \square P_k \) has a span of \( \sqrt{2} \), for some \( k \geq 2 \), for all \( m \geq 2 \). Furthermore, assume that there is a unit-distance representation of \( P_m \square P_k \) in \( \mathbb{R}^2 \) contained in a closed disk of diameter arbitrarily close to \( \sqrt{2} \) such that each copy of \( P_m \) in the representation is contained in a closed disk of diameter
arbitrarily close to 1. Let us consider a copy of $P_m$ which forms one side of the rectangular planar grid $P_m \Box P_k$. Let $D$ be a smallest closed disk that contains the representation of $P_m \Box P_k$ and assume that the diameter of $D$ is arbitrarily close to $\sqrt{2}$. Let the polygonal line $123 \cdots k$ be the representation of one side of the rectangular grid. This unit-distance representation of $P_m \Box P_k$ can be extended to a unit-distance representation of $P_m \Box P_{k+1}$ simply by adjoining to it a unit-distance representation of $P_2 \Box P_k$ in such a way that one longer side of the grid is isometric with $123 \cdots k$. We can prevent vertex overlapping because of the density property of $\mathbb{R}^2$.

It is quite natural to relate the span of a unit-distance graph to its graph-theoretic diameter $diam(G)$, defined to be the maximum among all graph-theoretic distances between two vertices in $G$.

**Theorem 2.3** For every real number $\epsilon > 0$, there exists a unit-distance graph $G$ in $\mathbb{R}^2$ such that $\frac{\text{span } G}{diam(G)} < \epsilon$.

**Proof**: Let $\epsilon > 0$ be given. Let $n > 1$ and consider the planar grid $P_n \times P_n$ whose graph-theoretic diameter is $2n - 2$ and whose span is $\sqrt{2}$ (by Lemma 2.1). By choosing $n$ to be sufficiently large, we can make the ratio $\frac{\sqrt{2}}{(2n-2)}$ less than $\epsilon$. \hfill $\square$

## 3 Extent of Unit-Distance Graphs in the Plane

The extent of a unit graph $G$ in $\mathbb{R}^2$ will be denoted by $\text{ext } G$. It is easy to see that $\text{ext } C_4 = 2$. In general, the extent of the cycle $C_n$ is given in the following theorem. The proof is easy and is omitted.

**Theorem 3.1** The extent of cycle $C_n$ of order $n \geq 3$ is given by

\[
\text{ext } C_n = \begin{cases} 
\frac{n}{2} & \text{if } n > 2 \text{ is even} \\
\frac{(n-1)^2}{2\sqrt{n(n-1)}} & \text{if } n \geq 3 \text{ is odd}.
\end{cases}
\]

For each $n \geq 1$, the $n$-cube, denoted by $Q_n$, is the graph $K_2 \times K_2 \times \cdots \times K_2$. By mathematical induction, one can show that $\text{ext } Q_n = n$. The planar grid $P_m \times P_n$ has extent equal to $m + n - 2$. Note that the diameter and the extent are numerically equal in the case of the planar grid and the $n$-cube.

**Theorem 3.2** For every real number $\epsilon > 0$, there exists a unit-distance graph $G$ in $\mathbb{R}^2$ such that $\frac{\text{ext } G}{diam(G)} < \epsilon$. 
Proof: Let \( \epsilon > 0 \). Choose a positive even integer \( k > 2 \) such that \( \frac{2k-1}{k^2-k+1} < \epsilon \) and construct the unit-distance graph \( G = S_k \) shown in Figure 2. The diameter of \( S_k \) is \( k + (k - 1)^2 \). On the other hand, the smallest closed disk containing the unit-distance representation of \( S_k \) has the line segment \( pq \) as a diameter. But \(|pq| = 2k - 1\); hence, \( \frac{\text{ext } G}{\text{diam } G} = \frac{2k-1}{k+(k-1)^2} = \frac{2k-1}{k^2-k+1} < \epsilon \). \( \square \)

![Figure 2: Unit-distance graph with extent less than the diameter.](image)

Let \( G \) be a unit-distance graph in \( \mathbb{R}^2 \). Since \( \text{span } G \leq \text{ext } G \), we have \( \frac{\text{span } G}{\text{diam } G} \leq \frac{\text{ext } G}{\text{diam } G} \). Thus, Theorem 2.3 follows from Theorem 3.2.

Theorem 3.2 states that the ratio of the extent to the diameter can be made arbitrarily small. Our next task is to determine an upper bound for the ratio of extent to the diameter over all unit-distance graphs in \( \mathbb{R}^2 \).

**Theorem 3.3** If \( G \) is a unit-distance graph in \( \mathbb{R}^2 \), then

\[
\frac{\text{span } G}{\text{diam } G} \leq \frac{2\sqrt{3}}{3}.
\]

Proof: Without loss of generality, assume that \( G \) is a connected graph. Let \( D \) be a disk having diameter equal to \( \text{ext } G \). Then \( G \) has a unit-distance representation contained in \( D \). Let \( \widehat{G} \) be a unit-distance representation of \( G \) in \( D \). If there are exactly two points (vertices) of \( \widehat{G} \) on the boundary of \( D \), say \( x \) and \( y \), then \( \text{ext } G \) is equal to the Euclidean distance between \( x \) and \( y \) which in turn is less that or equal to the graph theoretic distance between \( x \) and \( y \). But this graph theoretic distance does not exceed \( \text{diam } G \). Thus \( \text{ext } G \leq \text{diam } G \leq \frac{2\sqrt{3}}{3} \text{ diam } G \). Assume now that there are at least three points of \( \widehat{G} \) on the boundary of \( D \). Let \( a, b, c \) be three points \( \widehat{G} \) on the boundary of \( D \) and consider the triangle \( T \) whose vertices are these three points. Please refer to Figure 3.

Draw a diameter of \( D \) with one end point at \( a \) and let \( p \) be the other end point. Then either \( T \) lies on one side of the diameter as shown in (a) or the
We claim that angle $\angle bac < \pi$ in case (a). If $\angle c > \pi/2$, then all other points of $\hat{G}$ are above the side $ac$ where $b$ is, for it would be possible to contain $\hat{G}$ in a smaller closed disk. Thus, there must be points of $\hat{G}$ below the side $ac$. These points cannot all be interior to $D$ for it would again be possible to contain $\hat{G}$ in a smaller closed disk. Hence there exist points of $\hat{G}$ different from $a$ and $c$ that lie on the arc $apc$. Let $d$ be the point on the arc $apc$ that is closest to $c$ and consider the following cases.

Case 1. The point $d$ is the point $p$ itself. In this case, $\text{ext } G = |ap| \leq \text{diam } G \leq 2\sqrt{3} \text{ diam } G/3$.

Case 2. The point $d$ is such that $p$ is on the arc $cd$. In this case, the triangle $abd$ is like the triangle in case (c). Thus, it is enough to consider case (c).

Consider now the triangle $T$ shown in Figure 3(c). We may assume without loss of generality that there are no points of $\hat{G}$ in the arc $bpc$ except $b$ and $c$. We claim that angle $\angle bac < \pi/3$. Suppose that angle $\angle bac > \pi/3$, as shown in Figure 3(d). There are three subcases to consider. Assume that the angle $\angle abc$ is more than a right angle. Since there are no points of $\hat{G}$ on the arc $bpc$ except $b$ and $c$, then all other points of $\hat{G}$ to the right of the side $bc$ are interior points of $D$. It would then be possible to contain $\hat{G}$ in a smaller closed disk, and this is a contradiction. A second case is when the angle $\angle bac$ is a right angle. In this case, $\text{ext } G = |bc| \leq \text{diam } G \leq 2\sqrt{3} \text{ diam } G/3$. The third case is when the angle $\angle bac$ is less than a right angle (but greater than $\pi/3$). In this subcase, the center of $D$ is interior to the triangle $T$. Furthermore, one of the other two acute

Figure 3: Three points $a$, $b$, $c$ in $\hat{G}$ on the boundary of $D$. 

diameter cuts the side $bc$ of $T$, as shown in (c). Among all triangles falling under case (a), we choose one such that the angle $cap$ is minimum. In case (a), it is not possible that all other points of $\hat{G}$ are above the side $ac$ where $b$ is, for it would be possible to contain $\hat{G}$ in a smaller closed disk. Thus, there must be points of $\hat{G}$ below the side $ac$. These points cannot all be interior to $D$ for it would again be possible to contain $\hat{G}$ in a smaller closed disk. Hence there exist points of $\hat{G}$ different from $a$ and $c$ that lie on the arc $apc$. Let $d$ be the point on the arc $apc$ that is closest to $c$ and consider the following cases.

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angles of $T$ is less than $\frac{\pi}{3}$. Without loss of generality, assume that angle $bca$ is less that $\frac{\pi}{3}$. We can rename the vertices of $T$ and this reduces to case (c).

Everything now will be with reference to (c). The diameter $ap$ partitions the angle $bac$ into two angles and we may assume without loss of generality that the angle $pac$ is less that or equal to the angle $pab$. It follows that angle $pac$ is less that or equal to $\frac{\pi}{6}$. Now the triangle $pac$ is a right triangle with the right angle at $c$. By the Pythagorean theorem, we have $|ap|^2 = |ac|^2 + |cp|^2$. But $|ac| \leq d(a,c) \leq \operatorname{diam} G$ and $|cp| \leq \frac{1}{2} \operatorname{ext} G$. The first inequality follows from Lemma 1.1 while the second follows from the fact that angle $pac$ is at most $\pi/6$. Thus, we get $|ap|^2 \leq (\operatorname{diam} G)^2 + \frac{1}{4} (\operatorname{ext} G)^2$. Finally, since $|ap| = \operatorname{ext} G$, we get $(\operatorname{ext} G)^2 \leq (\operatorname{diam} G)^2 + \frac{1}{4}(\operatorname{ext} G)^2$. This last inequality yields the lemma. \hfill \Box

The bound given in Theorem 3.3 is best possible since it is attained by each of the graphs in the infinite family shown in Figure 4.

![Figure 4: Unit-distance graphs $G$ in the plane with $\operatorname{ext} G = \frac{2}{3} \sqrt{3}d$.](image)

We see from the above family of graphs that for each integer $d > 0$, there exists a unit-distance graph $G$ with diameter $d$ such that the extent of $G$ is equal to $\frac{2}{3} \sqrt{3}d$.

**Acknowledgements.** The authors would like to thank the Department of Science and Technology of the Philippines for partly funding this research.

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Received: December 28, 2015; Published: May 4, 2016