Simplicial Complexes and New Applications

V. Iorfida
University of Calabria, Campus of Arcavacata, Via P. Bucci
87036 – Rende, CS Italy

G. Restuccia
University of Messina, Dipartimento di Scienze Matematiche e Informatiche
Scienze Fisiche e Scienze della Terra, V.le F. Stagno D’Alcontres 31
98166 – Messina, Italy

Dedicated to Prof. Peter Gruber for his 75th birthday

Abstract
We propose three compositions of triangulations of sets of lattice points as basic configurations in the realization of design objects in different fields.

Keywords: Graded algebras, Gröbner Bases, Simplicial complexes, Triangulations

Introduction

There are various applications of Gröbner bases theory in many fields of mathematical modelling. In particular ranking and transportation problems can be studied using these methods ([1], [3], [5], [6]).

Another enthusiastic experiment is based on the creation of designs coming from the mathematical field [4]. Simple geometric figures have always achieved success in the realization of tissues, colored glasses, floorings and other objects, but it is more interesting to introduce, using computational algebra methods, fundamental cells for these objects that are very complicated from the mathematical point of view. They can be obtained as triangulations of sets of special lattice points coming from initial complexes of toric ideals in the polynomial ring \( \mathbb{Q}[x_1, \ldots, x_n] \) in \( n \) variables on the rational number field \( \mathbb{Q} \) ([6]).

In this note we show how to compose triangulations to obtain design objects in different fields as business or engineering.

In Section 1 we recall definitions and results on simplicial complexes, polyhedral
geometry, initial simplicial complexes with respect to a term order on the set of all monomials of the polynomial ring $Q[x_1,...,x_n]$ and some related results.

In Section 2 we consider triangulations of sets of lattice points of $\mathbb{N}^n$ or of $\mathbb{Z}^n$, where $\mathbb{N}$ is the semigroup of natural numbers and $\mathbb{Z}$ the group of integers, and we propose three fundamental cells given by triangulations and three possible compositions of them.

1. Algebraic and Geometric Tools

First we recall the following:

**Definition 1.1**
Let $I = \{1, 2, ..., n\}$. A simplicial complex $\Delta$ on $I$ is a collection of subsets of $I$ such that

1) $\emptyset \in \Delta$
2) $F \in \Delta \Rightarrow F' \in \Delta$, $\forall F' \subset F$.

Then $\Delta \subseteq P(I)$.
We can associate rings and ideals to each simplicial complex. Precisely:

1) $K[x_1,...,x_n] = S$ is the polynomial ring over a field $K$ in $n$ indeterminates, if $n$ is the number of the vertices of $\Delta$.

2) $I_\Delta$, the Stanley-Reisner ideal (a monomial ideal of $S$), defined as follows:

$$I_\Delta = \langle \{x_{i_1}...x_{i_m}\} s.t. \{i_1,...,i_m\} \notin \Delta \rangle, I_\Delta \subset K[x_1,...,x_n].$$

$I_\Delta$ is called the ideal of non-faces of $\Delta$, generated by all the monomials that identify the non-faces of $\Delta$.

3) The Stanley-Reisner ring (or face ring) of $\Delta$ with respect to $K$ is the ring

$$K[\Delta] = K[x_1,...,x_n]/I_\Delta.$$  

The monomials of $K[x_1,...,x_n]$ that do not vanish in $K[\Delta]$ are the only ones whose variables in the support form a face of $\Delta$.

In the following, we will adopt the notations in [6], chap. 2.

**Definition 1.3**
A polyhedron is a finite intersection of closed half-spaces in $\mathbb{R}^n$, with the usual topology of $\mathbb{R}^n$. A polyhedron can be written as follows:

$$P = \{x \in \mathbb{R}^n / A \cdot x \leq b; b \in \mathbb{R}^n\},$$

where $A$ is a matrix having $n$ columns with entries in $\mathbb{R}$.
Given $u_1, ..., u_m \in \mathbb{R}^n$, let us consider $P \subseteq \mathbb{R}^n$,

$$P = \text{pos}(\{u_1, ..., u_m\}) = \{\lambda_1 u_1 + ... + \lambda_m u_m / \lambda_1,..., \lambda_m \in \mathbb{R}^+\}.$$  

$P$ is called a polyhedral cone.

**Definition 1.4**
A set $B \subseteq \mathbb{R}^n$ is called a cone if for all $x, y \in B$, and $\alpha, \beta \in \mathbb{R}^+$, then $\alpha x + \beta y \in B$. 

Definition 1.5
A convex set in $\mathbb{R}^n$ is a set in which, for each pair of points, the segment that joins them is entirely contained in the set.

Definition 1.6
A convex hull $\text{conv}(Q)$ of a set $Q$ of $\mathbb{R}^n$ is the smallest convex set of $\mathbb{R}^n$ which contains $Q$. So $\text{conv}(Q)$ is also the intersection of all convex sets containing $Q$. In particular, if $Q$ consists of two points $v_1, v_2$, $\text{conv}\{v_1, v_2\}$ is the segment having endpoints $v_1, v_2$.

Definition 1.7
A polyhedral complex $\Delta$ is a finite collection of polyhedra in $\mathbb{R}^n$ satisfying the following conditions:

1. if $P \in \Delta$ and $F$ is a face of $P$, then $F \in \Delta$
2. if $P_1, P_2 \in \Delta$, then $P_1 \cap P_2$ is a face of $P_1$ and $P_2$.

Definition 1.8
A bounded polyhedron $Q$ is called a polytope. Each polytope $Q$ is the convex hull of a finite set of points $Q = \text{conv}\{v_1, ..., v_m\} = \left\{ \sum_{i=1}^{m} \lambda_i v_i / \lambda_i, ..., \lambda_m \in \mathbb{R}^+, \sum_{i=1}^{m} \lambda_i = 1 \right\}$.

Consider now the polynomial ring $S = K[x_1, x_2, ..., x_n]$, and an ideal $I \subset S$. Given an ordering on variables in $S$ and a term order on the monomials of $S$, we have the following facts:

1. if $\text{in}_<(I)$ is a squarefree (monomial) ideal, $\text{in}_<(I) = I_\Delta$ is the ideal of non-faces of a simplicial complex $\Delta$.
2. $\text{in}_<(I)$ is not necessarily squarefree.
3. If $I$ is any monomial ideal, then its radical ideal $\sqrt{I}$ is a squarefree monomial ideal.

Definition 1.9
Let $I \subset S$ be an ideal. The initial simplicial complex of $I$, denoted by $\Delta_<$, is the simplicial complex whose Stanley- Reisner ideal is $\sqrt{\text{in}_<(I)}$ (it depends from the chosen term order).

2. New Figures from the Triangulation of a Set of Lattice Points.

Let $A = \{a_1, ..., a_t\}$, where $a_i \in \mathbb{N}^d$, $i = 1, ..., t$. 
Definition 2.1

Let $\sigma$ be a subset of lattice points, $\sigma \subseteq A$ and $\text{pos}(\sigma)$ be the polyhedral cone generated by $\sigma$. A triangulation of $A$ is a collection $\Delta$ of subsets of $A$, such that the set $\{\text{pos}(\sigma), \sigma \in A\}$ is the set of cones of a simplicial fan whose support is $\text{pos}(A)$.

$$\text{supp}\ \{\text{pos}(\sigma), \sigma \in \Delta\} = \text{pos}(A),$$

where

$$\text{supp}(\Delta) = \bigcup_{\sigma \in \Delta} \text{pos}(\sigma)$$

Example 2.1

Let $A = \{(3, 0), (2, 1), (1, 2), (0, 3)\} \subset \mathbb{N}^2$. Put $(3,0) =1; (2,1) =2; (1,2) =3; (0,3) = 4$

The possible triangulations of $A$ are:

$\Delta^1 =\{\{1, 2\}, \{2, 3\}, \{3, 4\}\}; \Delta^2 =\{\{1, 2\}, \{2, 4\}\}; \Delta^3 =\{\{1, 3\}, \{3, 4\}\}; \Delta^4 =\{\{1\}\}$

Now we are going to propose some figures.

The first one is obtained from the lexicographic triangulation of a special set of lattice points, the second one is obtained from the reverse lexicographic triangulation, see figs. 2, and 5, of the same set of points.

Consider the semigroup under the cubic Veronese surface in $\mathbb{P}^9$

$$A = \{(i_1, i_2, i_3) \in \mathbb{N}^3 \mid i_1 + i_2 + i_3 = 3\}$$

We order the variables as follows:

$$x_1 > x_2 > \cdots > x_{10}.$$ 

We write the elements of $A$ lexicographically ordered:

$$A = \{(3,0,0), (2,1,0), (2,0,1), (1,2,0), (1,1,1), (1,0,2), (0,3,0), (0,2,1), (0,1,2), (0,0,3)\}$$

Consider the initial complex $\Delta_\prec$ given by $\text{in}_\prec I_A$, where $I_A$ is the toric ideal of $A$.

The triangulation of $\Delta_\prec$ looks like:
This triangulation will be the fundamental cell of our potential design object. If we introduce a coloring, we obtain.

The composition we propose is:

Fig.4 Image built by the lexicographic triangulation procedure
Now write the elements of $A$ in the reverse lexicographic order:

$$A = \{(3,0,0), (2,1,0), (1,2,0), (0,3,0), (2,0,1), (1,1,1)\}$$

The triangulation gives the fundamental cell:

Fig. 5- Reverse lexicographic triangulation.

Fig. 6 Colored structure with legend relatively to the reverse lexicographic triangulation

and some possible compositions are
For the last proposal, we consider the set

\[ B = \left\{ (-1, -1, 1), (-1, 0, 1), (-1, 1, 1), (0, -1, 1), (0, 0, 1), (0, 1, 1), \\
(1, 1, 1), (1, 0, 1), (1, 1, 1), (2, -1, 1), (2, 0, 1), (2, 1, 1) \right\} \]

that is the configuration of lattice points given in [6], pag.131, interesting for algebraic geometers. It represents the immersion of \( \mathbb{P}^1 \times \mathbb{P}^1 \) in \( \mathbb{P}^{11} \). The toric ideal \( I_B \) has a squarefree initial ideal, with respect to the lexicographic order, whose initial complex admits a triangulation illustrated by the following diagram ([6]). New sets of lattice points appear in [2], [3], where squarefree Veronese rings are studied.

Assuming it as fundamental cell, we can obtain interesting decorations or artistic works, as clothes or flooring borders.
The previous border can be doubled or triplicated, until we obtain attractive designs.

**Acknowledgements.** This work has been supported by National Group for Algebraic and Geometric Structure and their Applications (GNSAGA-INDAM).

**References**


Received: April 1, 2016; Published: May 12, 2016