Quasi-Bounded Supersolutions of Discrete Schrödinger Equations

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Abstract

In the framework of discrete potential theory on a strongly connected infinite network $X$, for the Schrödinger equation with positive potential $q(x)$, we obtain some properties of its positive $q$-solutions and positive $q$-supersolutions. We give also some conditions for the existence of positive singular $q$-harmonic solutions on $X$ which are necessarily unbounded; and a few properties of quasi-bounded $q$-supersolutions are also obtained.

Keywords: $q$-harmonic, $q$-superharmonic, quasi bounded $q$-harmonic function, Schrödinger equations

1 Introduction

Long back in the Euclidean case Brelot [3] initiated the study of solutions to the equation $\Delta (M) = c(M) u(M)$, $c(M) \geq 0$. After many years, Ozawa started considering this equation on Riemann surfaces. In the classification theory of
Riemann surfaces $\mathcal{R}$, the equation $\Delta u = Pu$ where $P(z)dx\,dy, P(z) \neq 0$ is a non-negative $C^1$-differential on $\mathcal{R}$, has been investigated extensively. Ozawa [6] initiated this study in $\mathcal{R}$ and continued it in [7] and [8]. Meanwhile Myrberg [4] showed that on every Riemann surface there always exists the Green function of $\Delta u = Pu$. Royden [10] studied the classification of Riemann surfaces by considering varied differential forms $P$ and Dirichlet integrals.

Earlier Parreau [9] had devised a method of classification of Riemann surfaces $\mathcal{R}$ by introducing two subclasses of positive harmonic functions on $\mathcal{R}$, namely quasi-bounded harmonic functions and singular harmonic functions. Taking the important study of Parreau’s in the context of the equation $\Delta u = Pu$ on $\mathcal{R}$, Ow [5] proves some properties of quasi-bounded and singular solutions of $\Delta u = Pu$ using the Wiener $P$-harmonic boundary.

It is of interest to mention here that Arsove and Leutwiler [2] give a general formulation of quasi-bounded and singular functions on bounded regions $\Omega$ in Euclidean space. Let $\mathcal{M}$ be the class of all nonnegative functions $u$ on $\Omega$ which are majorised by superharmonic functions. Using the notions of reduced functions and their lower semi-continuous regularisations an operator $S$ is defined on $\mathcal{M}$. A function $u$ in $\mathcal{M}$ is said to be quasi-bounded if $Su = 0$ and $u$ is called singular if $Su = u$.

A development based on the properties of the operator $S$ relates this study to that of Parreau’s on quasi-bounded and singular harmonic functions on Riemann surfaces.

In this note, we recast many of the above results on $\mathcal{R}$ in the framework of discrete potential theory on strongly connected infinite networks $X$. An equation of the form $\Delta u(x) = q(x)u(x)$ where $q(x) \geq 0$ is real-valued is referred to as a Schrödinger equation with potential $q(x)$. The properties of its positive $q$-solutions ($q(x)u(x) = \Delta u(x)$) and its positive $q$-supersolutions ($q(x)u(x) \geq \Delta u(x)$) are obtained; since it is possible that all the positive $q$-solutions are bounded for some $X$ (recall Schiff’s conjecture [11, p. 240] that there always exists a non-negative unbounded solution of $\Delta u = Pu$ on an open Riemann surface $\mathcal{R} \notin O_{PB}$), we give some conditions for the existence of positive singular $q$-harmonic functions on $X$ which are necessarily unbounded; and a few properties of quasi-bounded $q$-supersolutions are also obtained.

2 Preliminaries

Let $X$ be a set consisting of an infinitely countable set of points called vertices. Let $t(x,y)$ be a transition function on $X$. That is, $t : X \times X \to \mathbb{R}^+$ such that i) $t(x,y)$ and $t(y,x)$ are not necessarily equal, and ii) $t(x,x) = 0$ for all $x \in X$. If
also that given any pair of vertices \( x, y \) there are only a finite number of vertices for any vertex \( x \). We assume also that given any pair of vertices \( x, y \) such that \( t(x, y) > 0 \). Note that if \( y \sim x \) then it is not necessary \( x \sim y \). We assume that there are only a finite number of vertices for any vertex \( x \). We assume also that given any pair of vertices \( x, y \) there exists a finite sequence of vertices \( \{x = x_0, x_1, \ldots, x_n = y\} \) such that \( t(x_0, x_1)t(x_1, x_2)\ldots t(x_{n-1}, x_n) > 0 \). Thus \( \{X, t(x, y)\} \) is an infinite digraph that is strongly connected, locally finite and without self-loops.

For a subset \( E \) of \( X \), a vertex \( x \in E \) is said to be an interior vertex of \( E \) if all the neighbours of \( x \) in \( X \) are also in \( E \). The set of all the interior vertices of \( E \) is denoted by \( E \). For a real-valued function \( u(x) \) on \( E \), define the Laplace operator \( \Delta u(x) = \sum_{y \sim x} t(x, y)[u(y) − u(x)] \) for any \( x \in E \). If \( \Delta u(x) \leq 0 \) for every \( x \in E \), then \( u(x) \) is said to be superharmonic on \( E \), and if \( \Delta u(x) = 0 \) for every \( x \in E \) then \( u(x) \) is said to be harmonic on \( E \).

Let \( q(x) \geq 0 \) be defined on \( X \), \( q \not= 0 \). For a real-valued function \( v(x) \) on \( X \), if we define \( \Delta_q v(x) = \Delta v(x) − q(x)v(x) \), then \( \Delta_q \) is referred to as the Schrödinger operator with potential function \( q(x) \).

**Definition 2.1.** A real-valued function \( u(x) \) on a subset \( E \) is said to be Schrödinger harmonic (or \( q \)-harmonic) on \( E \) if \( \Delta_q u(x) = 0 \) for all \( x \in E \). If a real-valued function \( v(x) \) is the Schrödinger super solution on \( E \), that is \( \Delta_q v(x) \leq 0 \) for every \( x \in E \), then \( v(x) \) is said to be \( q \)-superharmonic on \( E \).

**Lemma 2.1** (1, Theorem 4.1.3). If \( f(x) \) is a real-valued function on \( X \), let \( F \) be the family of all \( q \)-superharmonic functions on \( X \) majorizing \( f \) on \( X \). If \( F \) is non-empty, then \( Rf(x) = \inf \{s(x) : s \in F\} \) is \( q \)-superharmonic on \( X \) and \( q \)-harmonic at each vertex \( a \) where \( f(a) \) is \( q \)-subharmonic.

As a consequence and with the help of Poisson modification we prove the following theorem as in [1, Theorem 4.1.1]:

**Theorem 2.2.** Suppose \( u \) is \( q \)-superharmonic and \( v \) is \( q \)-subharmonic on \( X \) such that \( u \geq v \). Then there exists a \( q \)-harmonic function \( h \) on \( X \) such that \( u \geq h \geq v \). This function \( h \) can be chosen such that if \( h' \) is another \( q \)-harmonic function such that \( u \geq h' \geq v \) then \( h \geq h' \).

In the above theorem, \( h \) is referred to as the greatest \( q \)-harmonic minorant of \( u \). If \( s \geq 0 \) is a \( q \)-superharmonic function whose greatest \( q \)-harmonic minorant is 0, then \( s \) is called a \( q \)-potential on \( X \).

If \( s(x) \) is the constant function 1, then \( \Delta_q s(x) \leq 0 \) for all \( x \in X \). Hence the constant function 1 is \( q \)-superharmonic but not \( q \)-harmonic on \( X \). Take
the greatest $q$-harmonic minorant $h(x)$ of 1 on $X$. If $h = 0$ that is 1 is a $q$-potential on $X$ then $X$ is said to be a $q$-parahyperbolic network. Otherwise $X$ is known as a $q$-bounded hyperbolic network. Thus $X$ is $q$-bounded hyperbolic network if and only if there exist bounded positive $q$-harmonic functions on $X$. Actually it is easy to check that $X$ is $q$-bounded hyperbolic if and only if there are non-zero bounded $q$-harmonic functions on $X$. We give below an example each for the two types of networks mentioned above.

**Example 1.** Let $X = \{\ldots, e_2, e_1, e_0, e_1, e_2, \ldots\}$ be an infinite linear graph with Markovian indices $p(e_n, e_{n+1}) = p(e_n, e_{n-1}) = \frac{1}{2}$ for all $n$. Let $q(x) \geq 0$ be defined on $X$ such that $q(e_n) = \frac{1}{4}$ for all $n$. Then any $q$-harmonic function $h(x)$ on $X$ is of the form $h(n) = A2^n + B2^{-n}$ where $A, B$ are constants. Hence the only bounded $q$-harmonic function on $X$ is 0. Consequently, $X$ is a $q$-parahyperbolic network.

**Example 2.** Let $X = \{e_0, e_1, e_2, \ldots\}$ be an infinite linear graph with the Markovian transition indices $p(e_0, e_1) = 1$ and for $n \geq 1$, $p(e_n, e_{n+1}) = \frac{2}{3}$ and $p(e_n, e_{n-1}) = \frac{1}{3}$. Let $q(x) \geq 0$ be the function on $X$ such that $q(e_0) = \frac{1}{3}$ and $q(e_n) = 0$ for $n \geq 1$. Then the function $h(x)$ defined on $X$ as $h(e_n) = 1 + \frac{1}{3} + \cdots + \frac{1}{3^n}$ for $n \geq 0$ is a bounded $q$-harmonic function on $X$. Hence $X$ is a $q$-bounded hyperbolic network. Note that in this example, any positive $q$-harmonic function on $X$ is bounded and proportional to $h(x)$.

### 3 Positive harmonic functions

The above examples show that there may or may not be any positive bounded $q$-harmonic functions on $X$. However since the positive constants are always $q$-superharmonic functions on $X$. There are always some positive $q$-harmonic functions on $X$ [1, Theorem 4.1.9]. Let $H_q^+$ denote the class of nonnegative c-harmonic functions on $X$. In this section, we rapidly go through the procedure to obtain an integral representation for positive $q$-harmonic functions on $X$, using the Choquet theorem, similar to the one given in [1, Section 3.2.3]. We start with a version of the classical Harnack property for the class of positive $q$-superharmonic functions on $X$.

**Lemma 3.1.** Let $S_q^+$ denote the class of positive $q$-superharmonic functions on $X$. Let $a, b$ be any two vertices in $X$. Then there exist two positive constants $A, B$ such that $As(b) \leq s(a) \leq Bs(b)$ for any $s \in S_q^+$.

**Proof.** Let $\{a = x_0, x_1, \ldots, x_k = b\}$ be a directed path connecting $a$ to $b$. Let $s \in S_q^+$. Since $\Delta_q s(a) \leq 0$, we have $[t(a) + q(a)] \geq \sum_{y \sim a} t(a, y)s(y)$; here $t(a) = \sum_{y \sim a} t(a, y)$. In particular, since $x_1 \sim a$, we have $[t(a) + q(a)] \geq t(a, x_1)s(x_1)$. Then starting with $\Delta_q s(x_1) \leq 0$, and proceeding similarly we
find that \([t(x_1) + q(x_1)]s(x_1) \geq t(x_1, x_2)s(x_2)\). A series of such inequalities leads to \(s(a) \geq \frac{t(a, x_1)}{t(a, x_1) + q(a)} \cdot \frac{t(x_1, x_2)}{t(x_1, x_2) + q(x_1)} \cdot \ldots \cdot \frac{t(x_n, b)}{t(x_n, b) + q(x_{n-1})} s(b)\), which is of the form \(s(a) \geq As(b)\). Similarly starting with a path from the vertex \(b\) to the vertex \(a\), we see that \(s(a) \leq Bs(b)\). □

Now by using Theorem 2.2, it is readily shown that \(H_q^+\) is a lattice for the natural order. Write \(E = H_q^+ - H_q^+\) and make \(E\) a topological vector space with the norm: if \(u, v \in E\), then define \(\|u - v\| = \sup |\frac{u(x) - v(x)}{1 + |u(x)| + |v(x)|}|\), \(x \in X\). Then \(E\) is a locally convex metrisable space. Let us fix a vertex \(x_0\) in \(X\) and consider a base \(B\) of the convex cone \(H_q^+\) defined by \(B = \{h \in H_q^+, h(x_0) = 1\}\). By using the Harnack property (Lemma 3.1) and the fact that \(X\) has only a countable number of vertices, it is readily shown that \(B\) is a compact base for the convex cone \(H_q^+\) in \(E\). Then the Choquet integral representation theorem \([\]\) leads to the following theorem. Recall that \(u \in H_q^+\) is said to be minimal if whenever \(v \in H_q^+\) and \(v \leq u\), then \(v\) is proportional to \(u\).

**Theorem 1.** Let \(h \in H_q^+\). Then there exists a unique positive measure \(\mu\) with support in the set \(\Lambda_1\) of minimal points in \(B\) such that \(h(x) = \int_{\Lambda_1} v(x) d\mu(v)\) for \(x \in X\).

### 4 Quasi-bounded q-harmonic functions

In the previous section we remarked that to any positive \(q\)-harmonic function \(u(x)\) is associated a unique measure \(\mu\) supported by \(\Lambda_1\). This minimal set \(\Lambda_1\) divides into two disjoint sets, the significance of which was investigated by Parreau [9] in the context of positive harmonic functions on a Riemann surface. In this section we consider the discrete analogues of Parreaus results with reference to positive \(q\)-harmonic functions on an infinite network.

**Definition 4.1.** A nonnegative harmonic function \(u\) on \(X\) is said to be a **quasi-bounded \(q\)-harmonic function** if and only if \(u\) is the increasing limit of bounded \(q\)-harmonic functions on \(X\); and a non-negative \(q\)-harmonic function \(v\) on \(X\) is said to be a **singular \(q\)-harmonic function** if and only if \(0\) is the only bounded non-negative \(q\)-harmonic function majorised by \(v\).

**Remark.** The question whether there exists any unbounded solution to the Schrödinger equation \(\Delta h(x) = q(x)h(x)\) will have a positive answer if there exists a singular \(q\)-harmonic function on \(X\). In a parahyperbolic network , any solution to the Schrödinger equation is unbounded and also is a singular harmonic function. Thus in a parahyperbolic network, there are unbounded solutions; this is the discrete analogue of a Myrberg result [4] which states that there always exists a non-negative unbounded solution of \(\Delta u = Pu\) on a
Riemann surface $\mathcal{R} \in O_{PB}$. The Example 1 in Section 2 describes a network in which singular harmonic functions exist, while Example 2 describes a case where there no unbounded $q$-harmonic functions on $X$.

**Proposition 4.1.** Let $h$ be a nonnegative $q$-harmonic function on $X$. Then $h = u + v$ where $u$ is a quasi-bounded $q$-harmonic function and $v$ is a singular $q$-harmonic function.

**Proof.** Let $\mathcal{B}$ be the class of all bounded positive $q$-harmonic functions $b(x)$ majorised by $h(x)$. Note $\mathcal{B}$ that is an increasingly ordered family. For if $b_1, b_2 \in \mathcal{B}$ then the least $q$-harmonic majorant of $\sup(b_1, b_2)$ is in $\mathcal{B}$ (Theorem 2.2). Since $X$ is a countable set, we can get an increasing sequence $b_n \in \mathcal{B}$ such that $\sup_n b_n(x) = \sup_{b \in \mathcal{B}} b(x)$ for every $x \in X$. As an increasing limit of bounded $q$-harmonic functions, the function $u(x)$ is a quasi-bounded $q$-harmonic function; also it is the largest quasi-bounded $q$-harmonic function majorised by $h(x)$.

Let $v(x) = h(x) - u(x)$. Then $v(x)$ is a singular $q$-harmonic function on $X$. For suppose a bounded non-negative $q$-harmonic function $b(x)$ is majorised by $v(x)$. Then $b(x) + u(x)$ is a quasi-bounded $q$-harmonic function majorised by $h(x)$; that is $b + u \in \mathcal{B}$. Hence by the construction of $u(x)$, we have $b + u \leq u$. That is $b = 0$ so that $v(x)$ a singular $q$-harmonic function on $X$. □

**Lemma 4.2.** Let $w$ be an upper-bounded $q$-subharmonic function majorised by a singular $q$-harmonic function on $X$. Then $w \leq 0$.

**Proof.** For $w^+$ is a bounded $q$-subharmonic function majorised by $v$. Hence there exists a bounded $q$-harmonic function $b$ such that $0 \leq w^+ \leq b \leq v$. Since $v$ is a singular $q$-harmonic function, $b = 0$. Hence $w \leq 0$. □

**Lemma 4.3.** Let $v_1, v_2$ be two singular $q$-harmonic functions on $X$. Then $v_1 + v_2$ is a singular $q$-harmonic function.

**Proof.** Let $b \geq 0$ be a bounded $q$-harmonic function such that $b \leq v_1 + v_2$. Then $b - v_2$ is an upper-bounded $q$-harmonic function majorised by $v_1$. Hence by Lemma 4.2, which shows that $b = 0$. □

**Lemma 4.4.** Let $h$ be singular $q$-harmonic and $u$ be quasi-bounded $q$-harmonic such that $0 \leq h \leq u$. Then $h = 0$

**Proof.** Write $u - h = u_1 + v_1$ where $u_1$ is a quasi-bounded $q$-harmonic function and $v_1$ is a singular $q$-harmonic function (Proposition 4.1). Then $u - u_1 = h + v_1 \geq 0$. Let $u = \lim b_n$. Then $b_n - u_1$ is an upper-bounded $q$-harmonic function majorised by $h + v_1$ which is a singular $q$-harmonic function (Lemma 4.3). Hence by Lemma 4.2, $b_n - u_1 \leq 0$ which implies $u = \lim b_n \leq u_1$. Consequently, $u = u_1$ so that $h + v_1 = 0$. That is $v_1 = h = 0$. □
Lemma 4.5. Let $h$ be $q$-harmonic and $u$ be quasi-bounded $q$-harmonic such that $0 \leq h \leq u$. Then $h$ is a quasi-bounded $q$-harmonic function.

Proof. Write $h = u_1 + v_1$ as the sum of a quasi-bounded $q$-harmonic function and a singular $q$-harmonic function. Then $v_1 \leq h \leq u$. Hence $v_1 = 0$ by Lemma 4.4. That is, $h$ is a quasi-bounded $q$-harmonic function on $X$. □

Proposition 4.6. Let $u$ be a quasi-bounded $q$-harmonic function and $v$ be a singular $q$-harmonic function. Then $s = \inf(u, v)$ is a $q$-potential on $X$.

Proof. Since $s$ is a nonnegative $q$-superharmonic function, $s = p + h$ is the sum of a $q$-potential $p$ and a non-negative $q$-harmonic function $h$. Since $h \leq u$, $h$ is a quasi-bounded $q$-harmonic function (Lemma 4.5); since $h \leq v$, $h$ is a singular $q$-harmonic function also. Hence $h = 0$ and $s = p$ is a $q$-potential on $X$. □

Remark. The above proposition can be interpreted as follows: Let $\mu, \nu$ be the measures representing $u, v$ respectively in their integral representations. Then $\mu$ and $\nu$ are mutually singular.

A $q$-harmonic function $h \geq 0$ is said to be a minimal $q$-harmonic function on $X$ if and only if for any $q$-harmonic function $w$ such that $0 \leq w \leq h$, we have $w = \mu u$ for some constant $0 \leq \mu \leq 1$.

Proposition 4.7. A minimal $q$-harmonic function $h$ is either a bounded $q$-harmonic function or a singular $q$-harmonic function.

Proof. Suppose $h$ is an unbounded minimal $q$-harmonic function. Let $b$ be a bounded $q$-harmonic function such that $0 \leq b \leq h$. Then $b = \mu h$, which is possible only if $b = 0$. Hence $h$ is a singular $q$-harmonic function. □

Remark. The above proposition is of interest in the context of describing the support of the measure on the minimal boundary $\Lambda_1$ representing a positive $q$-harmonic function in the Martin-Choquet integral representation.

Theorem 4.8. Let $h > 0$ be a $q$-harmonic function on $X$. Then there exist a quasi-bounded $q$-harmonic function $u$ and a singular $q$-harmonic function $v$ such that $h = u + v$. This decomposition is unique.

Proof. By Proposition 4.1, $h = u_1 + v_1$ where $u_1$ is the greatest quasi-bounded $q$-harmonic function majorised by $h$ and $v_1$ is a singular $q$-harmonic function on $X$. To prove the uniqueness, suppose $h = u_2 + v_2$ be another decomposition as the sum of a quasi-bounded $q$-harmonic function and a singular $q$-harmonic function.
Then \( v_2 - v_1 = u_2 - u_1 \geq 0 \) since is the greatest quasi-bounded \( q \)-harmonic function majorised by \( h \). Since the non-negative \( q \)-harmonic function \( v_2 - v_1 \) is majorised by the singular \( q \)-harmonic function \( v_2, v = v_2 - v_1 \) is a singular \( q \)-harmonic function. Now \( u_1 = \lim b_n \) where \( \{b_n\} \) is an increasing sequence of bounded \( q \)-harmonic functions. Hence \( b_n - u_2 \) is an upper bounded \( q \)-harmonic function majorised by \( v \). Hence by Lemma 4.2, \( b_n - u_2 \leq 0 \). Taking limit when \( n \to \infty \), we conclude \( u_1 - u_2 \leq 0 \). Consequently, \( u_1 = u_2 \) and the decomposition is unique. □

**Proposition 4.9.** Let \( h > 0 \) be a \( q \)-harmonic function on \( X \) with the unique decomposition \( h = u + v \). Then \( u \) is the greatest quasi-bounded \( q \)-harmonic function majorised by \( h \) and \( v \) is the greatest singular \( q \)-harmonic function majorised by \( h \).

**Proof.** The assertion concerning \( u \) has already been proved in Proposition 4.1. As for \( v \), let \( v_1 \) be a singular \( q \)-harmonic function such that \( v_1 \leq h = u + v \). Let \( u_2 + v_2 = u + v - v_1 \) be the decomposition with \( u_2 \) quasi-bounded \( q \)-harmonic and \( v_2 \) singular \( q \)-harmonic. Then by the uniqueness of decomposition, applied to the equation \( u_2 + (v_2 + v_1) = u + v \), we conclude \( u_2 = u \) and \( v_2 + v_1 = v \). In particular \( v_1 \leq v \), leading to the conclusion that \( v \) is the greatest singular \( q \)-harmonic minorant of \( h \). □

**Proposition 4.10.** Let \( \{u_n\} \) be an increasing sequence of quasi-bounded \( q \)-harmonic functions. If \( u(x) = \sup_n u_n(x) \) is finite at one vertex in \( X \), then \( u(x) \) is a quasi-bounded \( q \)-harmonic function on \( X \).

**Proof.** Since \( [t(x) + q(x)]u_n(x) = \sum_{y \sim x} t(x, y)u_n(y) \), taking limits we have \( [t(x) + q(x)]u(x) = \sum_{y \sim x} t(x, y)u(y) \). Hence if \( u(z) \) is finite at a vertex \( z \in X \), then \( u(y) \) is finite for all \( y \sim z \). This implies \( u(x) \) is finite for all \( x \) in \( X \), since we have assumed that \( X \) is strongly connected. Consequently, \( u(x) \) is a \( q \)-harmonic function on \( X \).

Write \( Q(x) = \sup_{b \in \mathcal{B}} b(x) \) where \( \mathcal{B} \) is the class of all bounded \( q \)-harmonic functions \( b(x) \) majorised by \( u(x) \); we have seen that \( Q(x) \) is a quasi-bounded \( q \)-harmonic function. Now for each \( n \), \( u_n = \sup_m b_{n,m} \) where \( b_{n,m} \) is a non-negative bounded \( q \)-harmonic function majorised by \( u_n \). Then for any fixed vertex \( z \) and \( \epsilon > 0 \), \( u(z) \leq u_n(z) + \epsilon \) for some \( n \). Also \( u_n(z) \leq b_{n,m}(z) + \epsilon \) for some \( m \). Consequently, \( u(z) \leq b_{n,m}(z) + 2\epsilon \leq Q(z) + 2\epsilon \). Since \( \epsilon \) is arbitrary, we conclude that \( u(z) \leq Q(z) \). Again the vertex \( z \) being arbitrary, we conclude that \( u \leq Q \) on \( X \). However by construction \( Q \leq u \). Thus \( u = Q \) is a quasi-bounded \( q \)-harmonic function on \( X \). □

**Corollary 4.1.** If \( \{u_n\} \) is a sequence of quasi-bounded \( q \)-harmonic functions and if \( u(x) = \sum_n u_n(x) \) is convergent at one vertex in \( X \), then \( u(x) \) is a quasi-bounded \( q \)-harmonic function on \( X \).
Proof. Since a finite sum of quasi-bounded $q$-harmonic functions is a quasi-bounded $q$-harmonic function, $v_n = \sum_{m=1}^{n} u_m$ is a quasi-bounded $q$-harmonic function and hence $u = \sup_n v_n$ is a quasi-bounded $q$-harmonic function on $X$. 

\[ \square \]

5 Quasi-bounded supersolutions of Schrödinger equations

A real-valued function $s$ on $X$ is said to be a supersolution of the Schrödinger equation with potential $q(x)$ if $q(x)s(x) \geq \Delta s(x)$. Actually we refer to this super-solution $s(x)$ as a superharmonic function on $X$. Let $s \geq 0$ be a $q$-superharmonic function on $X$. Then $s$ is the sum of a $q$-potential $p$ and a non-negative $q$-harmonic function $h$ on $X$. Let $h = u + v$ be the unique decomposition $h$ of as the sum of a quasi-bounded $q$-harmonic function $u$ and a singular $q$-harmonic function $v$. Thus every $q$-superharmonic function $s \geq 0$ is of the form $s = p + u + v$.

In this representation, $v$ is the greatest singular $q$-harmonic minorant of $s$. For suppose $v_1$ is a singular $q$-harmonic function such that $v_1 \leq s$. Then $v_1 - (u + v) \leq p$ which implies that $v_1 - (u + v) \leq 0$. Since $v_1 \leq u + v$, we have $v_1 \leq v$ (Proposition 4.8). Consequently, $v$ is the greatest singular $q$-harmonic minorant of $s$.

Notation. If $s \geq 0$ is a $q$-superharmonic function on $X$, then $J(s)$ denotes the greatest singular $q$-harmonic minorant of $s$.

Definition 5.1. A non-negative $q$-superharmonic function $s$ is said to be a quasi-bounded $q$-superharmonic function if $J(s) = 0$.

Remark.

1. If $p$ is any $q$-potential on $X$, then $J(p) = 0$.

2. Let $h \geq 0$ be a $q$-harmonic function on $X$. Then is quasi-bounded $q$-harmonic if and only if $J(h) = 0$; and $h$ is singular $q$-harmonic if and only if $J(h) = h$.

3. A non-negative $q$-superharmonic function $s$ is a quasi-bounded $q$-superharmonic function if and only if $s$ is of the form $s = p + u$ where $p$ is a $q$-potential and is $u$ a quasi-bounded $q$-harmonic function.

4. Any $q$-superharmonic function $s \geq 0$ is the unique sum of a quasi-bounded $q$-superharmonic function and a singular $q$-harmonic function.
5. If $s_1, s_2$ are two non-negative $q$-superharmonic functions on $X$ such that $s_1 \leq s_2$, then $J(s_1) \leq J(s_2)$.

**Lemma 5.1.** \{$v_n$\} be a sequence of singular $q$-harmonic functions on $X$. Let $v(x) = \sum_n v_n(x)$. If $v(x)$ is finite at one vertex, then $v(x)$ is a singular $q$-harmonic function on $X$.

**Proof.** If $v(x)$ is finite at one vertex, then $v(x)$ is real-valued and $q$-harmonic on the strongly connected network $X$. To show that $v(x)$ is singular $q$-harmonic, suppose $b$ is a bounded $q$-harmonic function such that $0 \leq b \leq v$ on $X$. Then $[b(x) - \sum_2^\infty v_n(x)]^+ \leq v_1(x)$. Hence, by Lemma 4.2, $b(x) \leq \sum_2^\infty v_n(x)$. Proceeding similarly we show that for any positive integer $m$, $b(x) \leq \sum_m^\infty v_n(x)$. Since $\sum_n v_n$ is convergent, we conclude that $b(x) = 0$ for any $x \in X$. This means that $v(x)$ is a singular $q$-harmonic function on $X$. $\square$

**Lemma 5.2.** Let \{$v_n$\} be an increasing sequence of singular $q$-harmonic functions such that $v(x) \lim_n v_n(x)$ is finite at some vertex in $X$. Then $v(x)$ is a singular $q$-harmonic function on $X$.

**Proof.** For any $n \geq 1$, write $u_n = v_n - v_{n-1}$, with $v_0 = 0$. Then $u_n$ is a nonnegative $q$-harmonic function majorised by a singular $q$-harmonic function, hence $u_n$ is a singular $q$-harmonic function on $X$. Since $v(x) = \lim_n v_n(x) = \sum_1^\infty u_n(x)$, by Lemma 5.1, $v(x)$ is a singular $q$-harmonic function on $X$. $\square$

**Lemma 5.3.** Let $\mathcal{C}$ be the set of all singular $q$-harmonic functions majorised by a non-negative $q$-superharmonic function $s$ on $X$. Then $J(s) = \sup_{v \in \mathcal{C}} v$.

**Proof.** Let $w(x) = \sup_{v \in \mathcal{C}} v(x)$. Note that $\mathcal{C}$ is an increasingly ordered family. For, if $v_1, v_2 \in \mathcal{C}$ then the least $q$-harmonic majorant $v_3$ of $\sup(v_1, v_2)$ is such that $v_3 \leq v_1 + v_2$. Since $v_1 + v_2$ is a singular $q$-harmonic function, $v_3$ is also singular $q$-harmonic and hence $v_3 \in \mathcal{C}$. Since $\mathcal{C}$ is increasingly ordered and $X$ is a countable set, there exists an increasing sequence $v_n \in \mathcal{C}$ such that $\lim_n v_n(x) = \sup_{v \in \mathcal{C}} v(x) = w(x)$. Hence by Lemma 5.2, $w(x)$ is a singular $q$-harmonic function and clearly it is the greatest singular $q$-harmonic minorant of $s(x)$. That is, $J(s) = \sup_{v \in \mathcal{C}} v$. $\square$

**Proposition 5.4.** Let \{$s_n$\} be a sequence of nonnegative $q$-superharmonic functions on $X$. Let $s(x) = \sum_n s_n(x)$ and $s(z)$ is finite at some vertex $z$. Then $s(x)$ is $q$-superharmonic on $X$ and $J(\sum_n s_n) = \sum_n J(s_n)$. $\square$

**Proof.** Let $s_n = p_n + u_n + v_n$ where $p_n$ is a $q$-potential, $u_n$ is a quasi-bounded $q$-superharmonic function and $v_n$ is a singular $q$-harmonic function. The assumption that $s(z)$ is finite implies that $p = \sum_n p_n$ is a $q$-potential, $u = \sum_n u_n$ is a quasi-bounded $q$-harmonic function (Corollary 4.1) and $v = \sum_n v_n$ is a singular $q$-harmonic function (Lemma 5.1) and $s = p + u + v$. Hence $J(s) = v = \sum_n v_n = \sum_n J(s_n)$. $\square$
Corollary 5.1. Let \(\{w_n\}\) be a sequence of quasi-bounded \(q\)-superharmonic function on \(X\). If \(\sum_n w_n(z)\) is finite for some vertex \(z \in X\), then \(w(x) = \sum_n w_n(x)\) is a quasi-bounded \(q\)-superharmonic function on \(X\).

Proof. Since \(J(w_n) = 0\) for each \(n\), by the above proposition \(J(w) = 0\). That is, \(w(x)\) is a quasi-bounded \(q\)-superharmonic function on \(X\). □

Proposition 5.5. Let \(s \geq 0\) be a \(q\)-superharmonic function on \(X\). Then \(s\) is a quasi-bounded \(q\)-superharmonic function if and only if \(\inf(s, v)\) is a \(q\)-potential for every singular \(q\)-harmonic function on \(X\).

Proof. Suppose \(s\) is a quasi-bounded \(q\)-harmonic function. Let \(h \geq 0\) be a \(q\)-harmonic function such that \(h \leq \inf(s, v)\) for a singular \(q\)-harmonic function \(v\). Then implies that \(h\) is a quasi-bounded \(q\)-harmonic function; further \(h \leq v\) implies that \(h = 0\). Hence \(\inf(s, v)\) is a \(q\)-potential.

On the other hand, suppose \(s \geq 0\) is a \(q\)-superharmonic function such that \(\inf(s, v)\) is a \(q\)-potential for any singular \(q\)-harmonic function \(v\). Now \(s = w + v_0\) where \(w\) is a quasi-bounded \(q\)-superharmonic function and \(v_0\) is a singular \(q\)-harmonic function. Since \(v_0 = \inf(s, v_0)\) where the right side is a \(q\)-potential by hypothesis and the left side is a non-negative \(q\)-harmonic function, we conclude \(v_0 = 0\) so that \(s = w\) is a quasi-bounded \(q\)-superharmonic function on \(X\). □

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