Performance Comparison of Relaxation Methods with Singular and Nonsingular Preconditioners for Singular Saddle Point Problems

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Abstract

In this paper, we first review the PU and Uzawa-SAOR relaxation methods with singular or nonsingular preconditioning matrices for solving singular saddle point problems, and then we provide numerical experiments to compare performance results of the relaxation iterative methods using nonsingular preconditioners with those using singular preconditioners.

Mathematics Subject Classification: 65F10, 65F50

Keywords: Relaxation iterative methods, Singular saddle point problem, Semi-convergence, Drazin inverse, Moore-Penrose inverse

1 Introduction

We consider the following large sparse augmented linear system

$$\begin{pmatrix} A & B \\ -B^T & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ -g \end{pmatrix},$$  

(1)

where $A \in \mathbb{R}^{m \times m}$ is a symmetric positive definite matrix, and $B \in \mathbb{R}^{m \times n}$ is a rank-deficient matrix with $m \geq n$. In this case, the coefficient matrix of (1) is singular and so the problem (1) is called a singular saddle point problem.
This type of problem appears in many different scientific applications, such as constrained optimization [10], the finite element approximation for solving the Navier-Stokes equation [6], the constrained least squares problems and generalized least squares problems [1, 13], and so on.


The purpose of this paper is to provide performance comparison results for relaxation iterative methods with singular or nonsingular preconditioning matrices for solving the singular saddle point problems (1). This paper is organized as follows. In Section 2, we provide preliminary results for semi-convergence of the basic iterative methods. In Section 3, we provide the PU and Uzawa-SAOR methods with nonsingular preconditioners. In Section 4, we provide the PU and Uzawa-SAOR methods with singular preconditioners. In Section 5, we provide numerical experiments to compare performance results of the PU and Uzawa-SAOR methods using nonsingular preconditioners with those using singular preconditioners. Lastly, some conclusions are drawn.

2 Preliminaries for Semi-convergence

For the coefficient matrix of the singular saddle point problem (1), we consider the following splitting

\[ A = \begin{pmatrix} A & B \\ -B^T & 0 \end{pmatrix} = D - L - U, \]

where

\[ D = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 0 \\ B^T & 0 \end{pmatrix}, \quad U = \begin{pmatrix} P - A & -B \\ 0 & Q \end{pmatrix}, \]

where \( P \in \mathbb{R}^{m \times m} \) is a SPD (Symmetric Positive Definite) matrix which approximates \( A \), and \( Q \in \mathbb{R}^{n \times n} \) is a SPD matrix or a singular symmetric positive semi-definite matrix which approximates the approximated Schur complement matrix \( B^T P^{-1} B \). Several authors have presented semi-convergence analysis for relaxation iterative methods corresponding to the splitting (2) and (3) such as
the PU method, PIU method, GSSOR method, the inexact Uzawa method, the Uzawa-SOR method, the Uzawa-AOR method, the Uzawa-SAOR method, GMSSOR method [2, 7, 5, 11, 12, 14, 16, 17, 18].

For simplicity of exposition, some notation and definitions are presented. For a Hermitian matrix $H$, $\lambda_{\min}(H)$ and $\lambda_{\max}(H)$ denote the minimum and maximum eigenvalues of $H$, respectively. For a square matrix $G$, $\mathcal{N}(G)$ denotes the null space of $G$ and $\rho(G)$ denotes the spectral radius of $G$. Let us recall some useful results on iterative methods for solving singular linear systems based on matrix splitting. For a matrix $E \in \mathbb{R}^{n \times n}$, the Drazin inverse [3] of $E$ is defined by the unique matrix $E^D$ which satisfies the following equations

$$E^D E E^D = E, \quad E^D E = E E^D, \quad E^{k+1} E^D = E^k,$$

where $k = \text{index}(E)$ which is the size of the largest Jordan block corresponding to the zero eigenvalue of $E$. Let $A = M - N$ be a splitting of a singular matrix $A$, where $M$ is nonsingular. Then an iterative method corresponding to this splitting for solving a singular linear system $Ax = b$ is given by

$$x_{i+1} = M^{-1} N x_i + M^{-1} b \quad \text{for } i = 0, 1, \ldots \quad (4)$$

Definition 2.1 The iterative method (4) is semi-convergent if for any initial guess $x_0$, the iteration sequence $\{x_i\}$ produced by (4) converges to a solution $x^*$ of the singular linear system $Ax = b$.

It is well-known that if $A$ is nonsingular, then the iterative method (4) is convergent if and only if $\rho(M^{-1}N) < 1$. Since $A$ is singular, the iteration matrix $M^{-1}N$ has an eigenvalue 1 and thus $\rho(M^{-1}N)$ can not be less than 1. Thus, we need to introduce its pseudo-spectral radius $\nu(M^{-1}N)$

$$\nu(M^{-1}N) = \max\{|\lambda| \mid \lambda \in \sigma(M^{-1}N) - \{1\} \}$$

where $\sigma(M^{-1}N)$ is the set of eigenvalues of $M^{-1}N$. Notice that a matrix $T$ is called semi-convergent if $\lim_{k \to \infty} T^k$ exists, or equivalently $\text{index}(I - T) = 1$ and $\nu(T) < 1$ [3].

Theorem 2.2 ([3]) The iterative method (4) is semi-convergent if and only if $\text{index}(I - M^{-1}N) = 1$ and $\nu(M^{-1}N) < 1$, i.e., $M^{-1}N$ is semi-convergent. In this case, the iteration sequence $\{x_i\}$ converges to the following $x^*$

$$x^* = (I - M^{-1}N)^D M^{-1} b + (I - (I - M^{-1}N)^D (I - M^{-1}N)) x_0.$$

The Moore-Penrose inverse [3] of a singular matrix $E \in \mathbb{R}^{n \times n}$ is defined by the unique matrix $E^\dagger$ which satisfies the following equations

$$E = EE^\dagger E, \quad E^\dagger = E^\dagger EE^\dagger, \quad (EE^\dagger)^T = EE^\dagger, \quad (E^\dagger E)^T = E^\dagger E.$$
Let \( A = M - N \) be a splitting of a singular matrix \( A \), where \( M \) is singular. Then an iterative method corresponding to this singular splitting for solving a singular linear system \( Ax = b \) is given by

\[
x_{i+1} = (I - M^\dagger A)x_i + M^\dagger b \text{ for } i = 0, 1, \ldots
\]  \hspace{1cm} (5)

As in Definition 2.1, the iterative method (5) is called semi-convergent if for any initial guess \( x_0 \), the iteration sequence \( \{x_i\} \) produced by (5) converges to a solution \( x^* \) of the singular linear system \( Ax = b \).

**Theorem 2.3** ([4]) \hspace{1cm} The iterative method (5) is semi-convergent if and only if \( \text{index}(M^\dagger A) = 1, \nu(I - M^\dagger A) < 1, \) and \( N(M^\dagger A) = N(A) \), i.e., \( I - M^\dagger A \) is semi-convergent and \( N(M^\dagger A) = N(A) \).

### 3 Relaxation methods with a nonsingular pre-conditioner \( Q \)

We first provide the Uzawa-SAOR method with nonsingular preconditioners for solving the singular saddle point problem (1) [11]. Let \( A \) be decomposed as \( A = D - L - U \), where \( D = \text{diag}(A) \) is the diagonal matrix, and \( L \) and \( U \) are strictly lower triangular and strictly upper triangular matrices, respectively. Let \( \omega, s \) and \( \tau \) be positive parameters. Then, the Uzawa-SAOR method with a SPD matrix \( Q \) can be written as

\[
\begin{pmatrix}
x_{k+1} \\
y_{k+1}
\end{pmatrix} = H_1(\omega, s, \tau) \begin{pmatrix}
x_k \\
y_k
\end{pmatrix} + M_1(\omega, s, \tau) \begin{pmatrix}
f \\
g
\end{pmatrix}, \quad k = 0, 1, 2, \ldots
\]  \hspace{1cm} (6)

where

\[
H_1(\omega, s, \tau) = \begin{pmatrix}
M(\omega, s) & 0 \\
-\tau BT & Q
\end{pmatrix}^{-1} \begin{pmatrix}
N(\omega, s) & -\omega B \\
0 & Q
\end{pmatrix},
\]

\[
M_1(\omega, s, \tau) = \begin{pmatrix}
M(\omega, s) & 0 \\
-\tau BT & Q
\end{pmatrix}^{-1} \begin{pmatrix}
\omega I & 0 \\
0 & \tau I
\end{pmatrix},
\]

\[
M(\omega, s) = (D - sL)C(\omega, s)^{-1}(D - sU),
\]

\[
N(\omega, s) = ((1 - \omega)D + (\omega - s)U + \omega L)C(\omega, s)^{-1}((1 - \omega)D + (\omega - s)L + \omega U),
\]

\[
C(\omega, s) = (2 - \omega)D + (\omega - s)(L + U).
\]  \hspace{1cm} (7)

Notice that \( C(\omega, s) \) is symmetric positive definite if \( 0 < \omega < s < 2 \) [11]. From (6) and (7), the Uzawa-SAOR method with a SPD matrix \( Q \) can be rewritten as

**Algorithm 1:** Uzawa-SAOR method with nonsingular \( Q \)

Choose \( \omega, s, \tau \), and initial vectors \( x_0, y_0 \)

\[
C(\omega, s) = (2 - \omega)D + (\omega - s)(L + U)
\]
For $k = 0, 1, \ldots$, until convergence
\begin{align*}
x_{k+1} &= x_k + \omega(D - sU)^{-1}C(\omega, s)(D - sL)^{-1}(f - Ax_k - By_k) \\
y_{k+1} &= y_k + \tau Q^{-1}(B^T x_{k+1} - g)
\end{align*}
End For

Liang and Zhang [9] showed the following semi-convergence results for the Uzawa-SAOR method with a nonsingular preconditioner $Q$ from convergence results of the Uzawa-SAOR method presented in [11].

**Theorem 3.1** Suppose that $Q$ is a SPD matrix. If $0 < \omega \leq s < 2$ and
\begin{equation*}
0 < \tau < \frac{2 \lambda_{\min}(M(\omega, s) + N(\omega, s))}{\omega(\rho(BQ^{-1}B^T))},
\end{equation*}
then $\nu(H_1(\omega, s, \tau)) < 1$ and thus the Uzawa-SAOR method is semi-convergent.

**Corollary 3.2** Suppose that $Q$ is a SPD matrix. If $0 < \omega \leq s < 2$ and
\begin{equation*}
0 < \tau < \frac{2 \lambda_{\min}(A)}{\rho(BQ^{-1}B^T)},
\end{equation*}
then the Uzawa-SAOR method is semi-convergent.

We next provide the PU (Parameterized Uzawa) method using nonsingular preconditioners for the singular saddle point problem which have been studied in [17]. Then, the PU method with a SPD matrix $Q$ can be written as

**Algorithm 2: PU Method with nonsingular $Q$**

Choose $\omega$, $\tau$ and initial vectors $x_0$, $y_0$
For $k = 0, 1, \ldots$, until convergence
\begin{align*}
x_{k+1} &= (1 - \omega)x_k + \omega A^{-1}(f - By_k) \\
y_{k+1} &= y_k + \tau Q^{-1}(B^T x_{k+1} - g)
\end{align*}
End For

where $\omega > 0$ and $\tau > 0$ are relaxation parameters. Zheng et al [17] proved the following theorem about optimal parameters and the corresponding optimal semi-convergence factor for the PU method with a nonsingular $Q$.

**Theorem 3.3** Suppose that $Q$ is a SPD matrix. Let $\mu_{\min}$ and $\mu_{\max}$ be the smallest and largest nonzero eigenvalues of the matrix $Q^{-1}B^TA^{-1}B$, respectively. Then the optimal parameters $\omega_o$ and $\tau_o$ are given by
\begin{align*}
\omega_o &= \frac{4\sqrt{\mu_{\min}\mu_{\max}}}{(\sqrt{\mu_{\min}} + \sqrt{\mu_{\max}})^2} \quad \text{and} \quad \tau_o = \frac{1}{\sqrt{\mu_{\min}\mu_{\max}}},
\end{align*}
and the corresponding optimal semi-convergence factor is given by
\begin{equation*}
\frac{\sqrt{\mu_{\max}} - \sqrt{\mu_{\min}}}{\sqrt{\mu_{\max}} + \sqrt{\mu_{\min}}}.
\end{equation*}
4 Relaxation methods with a singular preconditioner $Q$

We first provide the Uzawa-SAOR method with singular preconditioners for solving the singular saddle point problem (1). Let $Q$ be chosen as $Q = B^T M^{-1} B$, where $M$ is a SPD matrix which approximates $A$. Then the Uzawa-SAOR method with the singular matrix $Q$ is defined by

$$
\begin{pmatrix}
  x_{k+1} \\
  y_{k+1}
\end{pmatrix} = H_3(\omega, s, \tau) \begin{pmatrix}
  x_k \\
  y_k
\end{pmatrix} + M_3(\omega, s, \tau) \begin{pmatrix}
  f \\
  -g
\end{pmatrix}, \quad k = 0, 1, 2, \ldots, \tag{8}
$$

where

$$
H_3(\omega, s, \tau) = I - M(\omega, s, \tau)^\dagger \Omega A
$$

$$
M_3(\omega, s, \tau) = M(\omega, s, \tau)^\dagger \Omega
$$

$$
M(\omega, s, \tau) = \begin{pmatrix}
  M(\omega, s) & 0 \\
  -\tau B^T & Q
\end{pmatrix}
$$

$M(\omega, s)$ and $N(\omega, s)$ are defined as in (7).

Since $QQ^\dagger B^T = B^T$, simple calculation yields

$$
M(\omega, s, \tau)^\dagger = \begin{pmatrix}
  M(\omega, s)^{-1} & 0 \\
  \tau Q^\dagger B^T M(\omega, s)^{-1} & Q^\dagger
\end{pmatrix}
$$

(9)

and

$$
H_3(\omega, s, \tau) = \begin{pmatrix}
  I_m - \omega M(\omega, s)^{-1} A & -\omega M(\omega, s)^{-1} B \\
  -\omega \tau Q^\dagger B^T M(\omega, s)^{-1} A + \tau Q^\dagger B^T & I_n - \omega \tau Q^\dagger B^T M(\omega, s)^{-1} B
\end{pmatrix}.
$$

(10)

From (8), (9) and (10), the Uzawa-SAOR method with the singular matrix $Q$ can be rewritten as

**Algorithm 3: Uzawa-SAOR method with singular $Q$**

Choose $\omega$, $s$, $\tau$, and initial vectors $x_0$, $y_0$

$C(\omega, s) = (2 - \omega) D + (\omega - s)(L + U)$

For $k = 0, 1, \ldots$, until convergence

$x_{k+1} = x_k + \omega (D - sU)^{-1} C(\omega, s)(D - sL)^{-1} (f - Ax_k - By_k)$

$y_{k+1} = y_k + \tau Q^\dagger (B^T x_{k+1} - g)$

End For

Here, $Q^\dagger$ (Moore-Penrose inverse of the matrix $Q$) is computed only once to reduce computational amount, and then it is stored for later use.

Liang and Zhang [9] also showed the following semi-convergence results for the Uzawa-SAOR method with the singular preconditioner $Q$ from convergence results of the Uzawa-SAOR method presented in [11].
Theorem 4.1 Let $Q$ be chosen as $Q = B^T M^{-1} B$, where $M$ is a SPD matrix which approximates $A$. Then the Uzawa-SAOR method for solving the singular saddle point problem (1) is semi-convergent if $0 < \omega \leq s < 2$ and

$$0 < \tau < \frac{2 \lambda_{\min}(M(\omega, s) + N(\omega, s))}{\omega \rho(BQ^TB)}.$$

Corollary 4.2 Let $Q$ be chosen as $Q = B^T M^{-1} B$, where $M$ is a SPD matrix which approximates $A$. If $0 < \omega \leq s < 2$ and

$$0 < \tau < \frac{2 \lambda_{\min}(A)}{\rho(BQ^TB)},$$

then the Uzawa-SAOR method for solving the singular saddle point problem (1) is semi-convergent.

We next provide the PU method with the singular preconditioner $Q$ for solving the singular saddle point problem (1), which can be written as

Algorithm 4: PU Method with singular $Q$

Choose $\omega$, $\tau$ and initial vectors $x_0$, $y_0$.
For $k = 0, 1, \ldots$, until convergence

$$x_{k+1} = (1 - \omega)x_k + \omega A^{-1}(f - By_k)$$
$$y_{k+1} = y_k + \tau Q^T(B^T x_{k+1} - g)$$

End For

where $\omega > 0$ and $\tau > 0$ are relaxation parameters. From Theorems 2.3 and 3.3, we can easily obtain the following theorem about the optimal parameters and the corresponding optimal semi-convergence factor for the PU method with the singular matrix $Q$.

Theorem 4.3 Let $Q$ be chosen as $Q = B^T M^{-1} B$, where $M$ is a SPD matrix which approximates $A$, and let $\mu_{\min}$ and $\mu_{\max}$ be the smallest and largest nonzero eigenvalues of the matrix $Q^TB^TA^{-1}B$, respectively. Then the optimal parameters $\omega_o$ and $\tau_o$ are given by

$$\omega_o = \frac{4\sqrt{\mu_{\min}\mu_{\max}}}{(\sqrt{\mu_{\min}} + \sqrt{\mu_{\max}})^2} \quad \text{and} \quad \tau_o = \frac{1}{\sqrt{\mu_{\min}\mu_{\max}}}$$

and the corresponding optimal semi-convergence factor is given by

$$\frac{\sqrt{\mu_{\max}} - \sqrt{\mu_{\min}}}{\sqrt{\mu_{\max}} + \sqrt{\mu_{\min}}}.$$
5 Numerical results

In this section, we provide numerical experiments to compare performance results of the PU and Uzawa-SAOR methods using nonsingular preconditioners with those using singular preconditioners for solving the singular saddle point problem (1). In Tables 3 and 4, Iter denotes the number of iteration steps and CPU denotes the elapsed CPU time in seconds. In Table 4, CPU_1 denotes the elapsed CPU time excluding the computational time for Q^\dagger.

Example 5.1 ([17]) We consider the saddle point problem (1), in which

\[
A = \begin{pmatrix}
I \otimes T + T \otimes I & 0 \\
0 & I \otimes T + T \otimes I
\end{pmatrix} \in \mathbb{R}^{2p^2 \times 2p^2},
\]

\[
B = \begin{pmatrix}
\hat{B} & b_1 \\
\tilde{B} & b_2
\end{pmatrix} \in \mathbb{R}^{2p^2 \times (p^2 + 2)},
\]

\[
\hat{B} = \begin{pmatrix}
F \\
0
\end{pmatrix},
\]

\[
b_1 = \hat{B} \begin{pmatrix}
e_{p^2/2} \\
0
\end{pmatrix},
\]

\[
b_2 = \hat{B} \begin{pmatrix}
e_{p^2/2} \\
0
\end{pmatrix},
\]

\[
e_{p^2/2} = (1, 1, \ldots, 1)^T \in \mathbb{R}^{p^2/2},
\]

\[
T = \frac{1}{h^2} \cdot \text{tridiag}(-1, 2, -1) \in \mathbb{R}^{p \times p},
\]

\[
F = \frac{1}{h} \cdot \text{tridiag}(-1, 1, 0) \in \mathbb{R}^{p \times p},
\]

with \( \otimes \) denoting the Kronecker product and \( h = \frac{1}{p+1} \) the discretization mesh size. For this example, \( m = 2p^2 \) and \( n = p^2 + 2 \). Thus the total number of variables is \( 3p^2 + 2 \). We choose the right hand side vector \((f^T, -g^T)^T\) such that the exact solution of the saddle point problem (1) is \((x^*_T, y^*_T)^T = (1, 1, \ldots, 1)^T \in \mathbb{R}^{m+n}\). Numerical results for this example are listed in Tables 3 and 4.

For the nonsingular case of \( Q \), the preconditioning matrices \( Q \) are chosen as in Table 1, where \( \hat{Q} \) denotes a block diagonal matrix consisting of two submatrices \( \hat{B}^T \hat{A}^{-1} \hat{B} \) and \( \tilde{B}^T \tilde{B} \). For the singular case of \( Q \), the preconditioning matrices \( Q \) are chosen as in Table 2. Notice that \( Q^\dagger \) is computed only once using the Matlab function \( \text{pinv} \) with a drop tolerance of \( 10^{-13} \), and then it is stored for later use.

In all experiments, the initial vector was set to the zero vector. Let \( \| \cdot \| \) denote the \( L_2 \)-norm. The iterations for the relaxation iterative methods are terminated if the current iteration satisfies \( \text{RES} < 10^{-6} \), where \( \text{RES} \) is defined by

\[
\text{RES} = \frac{\sqrt{\|f - Ax_k - By_k\|^2 + \|g - B^T x_k\|^2}}{\sqrt{\|f\|^2 + \|g\|^2}}.
\]

All numerical tests are carried out on a PC equipped with Intel Core i5-4570 3.2GHz CPU and 8GB RAM using MATLAB R2013b. In Tables 3 and 4, we report the numerical results for two different values of \( m \) and \( n \) and six cases for the matrix \( Q \). Relaxation iterative methods proposed in this paper...
depend on the parameters to be used. For the PU method, the parameters $\omega$ and $\tau$ are chosen as the optimal parameters stated in Theorems 3.3 and 4.3. For the Uzawa-SAOR method, the parameters are not optimal and are chosen as the best one by tries.

**Table 1. Choices of the nonsingular matrix $Q$ with $\hat{Q} = \text{Diag}(\hat{B}^T A^{-1} \hat{B}, \hat{B}^T \hat{B})$**

<table>
<thead>
<tr>
<th>Case Number</th>
<th>$Q$ Description</th>
<th>$\hat{A} = \text{diag}(A)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>$Q$</td>
<td>$A = \text{tridiag}(A)$</td>
</tr>
<tr>
<td>II</td>
<td>$\hat{Q}$</td>
<td>$\hat{A} = \text{tridiag}(A)$</td>
</tr>
<tr>
<td>III</td>
<td>tridiag($\hat{Q}$)</td>
<td>$\hat{A} = \text{tridiag}(A)$</td>
</tr>
<tr>
<td>IV</td>
<td>tridiag($\hat{Q}$)</td>
<td>$\hat{A} = A$</td>
</tr>
</tbody>
</table>

**Table 2. Choices of the singular matrix $Q$**

<table>
<thead>
<tr>
<th>Case Number</th>
<th>$Q$ Description</th>
<th>$M = \text{diag}(A)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>V</td>
<td>$B^T M^{-1} B$</td>
<td>$\hat{M} = \text{tridiag}(A)$</td>
</tr>
<tr>
<td>VI</td>
<td>$B^T M^{-1} B$</td>
<td>$\hat{M} = \text{tridiag}(A)$</td>
</tr>
</tbody>
</table>

**Table 3. Performance of PU and Uzawa-SAOR methods with nonsingular $Q$**

<table>
<thead>
<tr>
<th>$m$</th>
<th>$n$</th>
<th>$Q$</th>
<th>$\omega$</th>
<th>$\tau$</th>
<th>Iter</th>
<th>$CPU$</th>
<th>$\omega$</th>
<th>$s$</th>
<th>$\tau$</th>
<th>Iter</th>
<th>$CPU$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1152</td>
<td>578</td>
<td>I</td>
<td>0.2489</td>
<td>0.1423</td>
<td>131</td>
<td>0.175</td>
<td>0.90</td>
<td>1.58</td>
<td>0.50</td>
<td>107</td>
<td>0.060</td>
</tr>
<tr>
<td></td>
<td></td>
<td>II</td>
<td>0.3307</td>
<td>0.1985</td>
<td>90</td>
<td>0.232</td>
<td>0.90</td>
<td>1.55</td>
<td>1.00</td>
<td>105</td>
<td>0.182</td>
</tr>
<tr>
<td></td>
<td></td>
<td>III</td>
<td>0.5622</td>
<td>2.9447</td>
<td>44</td>
<td>0.048</td>
<td>0.85</td>
<td>1.59</td>
<td>1.40</td>
<td>98</td>
<td>0.024</td>
</tr>
<tr>
<td></td>
<td></td>
<td>IV</td>
<td>0.6199</td>
<td>3.3734</td>
<td>37</td>
<td>0.042</td>
<td>0.86</td>
<td>1.59</td>
<td>1.35</td>
<td>95</td>
<td>0.023</td>
</tr>
<tr>
<td>2048</td>
<td>1026</td>
<td>I</td>
<td>0.1956</td>
<td>0.1084</td>
<td>174</td>
<td>0.415</td>
<td>0.93</td>
<td>1.57</td>
<td>0.48</td>
<td>150</td>
<td>0.147</td>
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<tr>
<td></td>
<td></td>
<td>II</td>
<td>0.2635</td>
<td>0.1519</td>
<td>120</td>
<td>0.701</td>
<td>0.90</td>
<td>1.55</td>
<td>1.00</td>
<td>156</td>
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<td>III</td>
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<td>0.099</td>
<td>0.85</td>
<td>1.60</td>
<td>1.42</td>
<td>132</td>
<td>0.048</td>
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<tr>
<td></td>
<td></td>
<td>IV</td>
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<td>43</td>
<td>0.082</td>
<td>0.86</td>
<td>1.59</td>
<td>1.40</td>
<td>124</td>
<td>0.048</td>
</tr>
</tbody>
</table>

The relaxation iterative methods with nonsingular preconditioner $Q$ performs better than those with singular $Q$ (see Tables 3 and 4). The reason is as follows: For nonsingular $Q$, $Q^{-1}b$ is computed using Cholesky factorization of $Q$ without constructing $Q^{-1}$ explicitly, so computational cost is cheap. For singular $Q$, $Q^\dagger b$ is computed using matrix-times-vector operation after constructing $Q^\dagger$ explicitly, which is very time-consuming. Note that $Q^\dagger$ is constructed using the singular value decomposition of $Q$, which requires a lot of computational amount. If we exclude the computational time for constructing $Q^\dagger$, then relaxation iterative methods with singular $Q$ are comparable to those with nonsingular $Q$ (see $CPU$ in Table 3 and $CPU_1$ in Table 4).
Table 4. Performance of PU and Uzawa-SAOR methods with singular $Q$

<table>
<thead>
<tr>
<th>m</th>
<th>n</th>
<th>$Q$</th>
<th>$\omega$</th>
<th>$s$</th>
<th>$\tau$</th>
<th>Iter</th>
<th>CPU</th>
<th>CPU</th>
<th>PU</th>
<th>Uzawa-SAOR</th>
</tr>
</thead>
<tbody>
<tr>
<td>1152</td>
<td>578</td>
<td>V</td>
<td>0.2499</td>
<td>0.1423</td>
<td>131</td>
<td>0.214</td>
<td>0.129</td>
<td>0.90</td>
<td>1.58</td>
<td>0.50</td>
</tr>
<tr>
<td></td>
<td></td>
<td>VI</td>
<td>0.3307</td>
<td>0.1985</td>
<td>90</td>
<td>0.175</td>
<td>0.090</td>
<td>0.90</td>
<td>1.55</td>
<td>1.00</td>
</tr>
<tr>
<td>2048</td>
<td>1026</td>
<td>V</td>
<td>0.1956</td>
<td>0.1084</td>
<td>174</td>
<td>0.767</td>
<td>0.362</td>
<td>0.93</td>
<td>1.58</td>
<td>0.47</td>
</tr>
<tr>
<td></td>
<td></td>
<td>VI</td>
<td>0.2635</td>
<td>0.1519</td>
<td>120</td>
<td>0.657</td>
<td>0.252</td>
<td>0.90</td>
<td>1.55</td>
<td>1.00</td>
</tr>
</tbody>
</table>

For both nonsingular and singular matrix of $Q$, Uzawa-SAOR method with parameters chosen by tries performs much better than PU method with optimal parameters. However, we do not have a formula for finding optimal or near optimal parameters of Uzawa-SAOR method, which should be done in the future work.

6 Conclusions

We have provided performance comparison results for relaxation iterative methods with singular or nonsingular preconditioners for solving the singular saddle point problem (1). Numerical experiments showed that relaxation iterative methods with nonsingular preconditioning matrix $Q$ performs better than those with singular $Q$. The reason is that relaxation iterative methods with singular $Q$ takes a lot of CPU time for constructing $Q^\dagger$. If we exclude the computational time for constructing $Q^\dagger$, then relaxation iterative methods with singular $Q$ are comparable to those with nonsingular $Q$.

For both singular and nonsingular matrix of $Q$, Uzawa-SAOR method with parameters chosen by tries performs much better than PU method with optimal parameters. So, the problem of finding optimal or near optimal parameters of Uzawa-SAOR method will be a good challenging problem for the future work.

For singular preconditioning matrix $Q$, we only considered the form of $Q = B^T M^{-1} B$, where $M$ is a SPD matrix which approximates $A$, which restricts the choices of $Q$. Further research is needed to study semi-convergence analysis for other forms of singular matrix $Q$, so that we can try many different kinds of $Q$ in order to achieve the best possible performance of relaxation iterative methods.

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References


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