Generalized Nash Equilibria in Riemannian Manifolds

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Abstract

In this paper, by using Begle’s fixed point theorem for compact acyclic multimaps, we will prove a basic existence theorem of Nash equilibrium for generalized game \( \mathcal{G} = (X_i; T_i, f_i)_{i \in I} \) with geodesic convex values in a finite dimensional Riemannian manifold.

Mathematics Subject Classification: 91A10, 58B20, 47H10

Keywords: Nash equilibrium; geodesic convex; acyclic ANR; Riemannian manifold

1 Introduction

In 1950, Nash [8] established the pioneering result on the existence of equilibrium for abstract economies. Since then, the classical results of Nash [8] and Debreu [3] have served as basic references for the existence of Nash equilibrium for generalized games. Till now, there have been a number of generalizations, and also many applications of those theorems have been found in several areas, e.g., see [4-7,9] and references therein.

Existence results of Nash equilibria are often derived from fixed point theorems, minimax/variational inequalities or KKM-type intersection theorems, e.g., see [4,7,9] and the references therein. Recently, there have been some progress on the existence of Nash equilibrium without using the convexity
or even more, when the domains are not convex in the usual sense. In [5], by embedding the strategy sets into suitable Riemannian manifolds, Kristály proved the existence theorem of a Nash equilibrium for the noncooperative game $G = (X_i; f_i)_{i \in I}$ of normal form by applying the McClendon variational inequality [6] with the acyclic values in a compact finite dimensional ANR. Next, Kristály proved various applications of variational problems and equilibrium problems in Riemannian manifolds as in [6]. However, till now, via Debreu’s method in [3], there does not exist a generalization of Kristaly’s theorem into generalized games with constraint correspondences in Riemannian manifold settings.

In this paper, using Begle’s fixed point theorem [1] for compact acyclic multimaps, we establish the existence theorem of Nash equilibrium for the generalized game $G = (X_i; T_i, f_i)_{i \in I}$ of normal form with geodesic convex values in a finite dimensional Riemannian manifold.

2 Preliminaries

We begin with some notations and definitions. If $A$ is a nonempty set, we shall denote by $2^A$ the family of all subsets of $A$. Let $E$ be a topological vector space and $X$ be a nonempty subset of $E$. If $T : X \to 2^E$ and $S : X \to 2^E$ are multimaps (or correspondences), then $S \cap T : X \to 2^E$ is a correspondence defined by $(S \cap T)(x) = S(x) \cap T(x)$ for each $x \in E$. A multimap $T$ is closed if the graph $Gr T := \{(x, y) \mid y \in T(x) \text{ for each } x \in X\}$ is closed in $X \times E$.

Let $X$ be a topological space and $f : X \to \mathbb{R}$ be a real-valued function. Recall that $f$ is lower semicontinuous if for each $t \in \mathbb{R}$, $\{x \in X \mid f(x) \leq t\}$ is closed in $X$. When $f$ is lower semicontinuous, if $(x_n) \to \bar{x}$ and $y_n \in f(x_n)$ for each $n$, then $f(\bar{x}) \leq \lim \inf_{n \to \infty} f(x_n)$. A function $f$ is upper semicontinuous if $-f$ is lower semicontinuous, and $f$ is said to be continuous if $f$ is both lower semicontinuous and upper semicontinuous.

Let $I = \{1, 2, \ldots, n\}$ be a finite (or possibly countably infinite) set of players. For each $i \in I$, $X_i$ is a non-empty topological space as an action space, and denote $X_{-i} := \Pi_{j \in I - \{i\}} X_j$. For an action profile $x = (x_1, \ldots, x_n) \in X = \Pi_{i \in I} X_i$, we shall write $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in X_{-i}$, and we may simply write $x = (x_{-i}, x_i) \in X_{-i} \times X_i = X$.

Recall that a nonempty topological space is acyclic if all of its reduced Čech homology groups over rationals vanish. For nonempty subsets in a topological vector space, the following implication is well-known that

convex $\Rightarrow$ star-shaped $\Rightarrow$ contractible $\Rightarrow$ acyclic $\Rightarrow$ connected, and not conversely, e.g., see [9].

Next, we recall some notions and terminologies on the generalized Nash
equilibrium for pure strategic games as in [4,5,9]. Let \( I = \{1, 2, \ldots, n\} \) be a finite (or possibly countably infinite) set of players. A noncooperative \textit{generalized game of normal form} is an ordered \( 3n \)-tuple \( \mathcal{G} = (X_i; T_i, f_i)_{i \in I} \) where for each player \( i \in I \), \( X_i \) is a pure strategy space for the player \( i \), and the set \( X := \prod_{i=1}^{n} X_i = X_{-i} \times X_i \), \textit{joint strategy space}, is the Cartesian product of the individual strategy spaces, and the element of \( X_i \) is called a \textit{strategy}. And, \( f_i : \mathbb{R} \rightarrow \mathbb{R} \) is a \textit{payoff function} (or \textit{utility function}), and \( T_i : X \rightarrow 2^{X_i} \) is a \textit{constraint correspondence} for the player \( i \).

Next, we recall the following which generalizes the Nash equilibrium by adopting the constraint correspondences:

**Definition 2.1.** Let \( \mathcal{G} = (X_i; T_i, f_i)_{i \in I} \) be a noncooperative generalized game of normal form. Then, a strategy \( n \)-tuple \( \bar{x} = (\bar{x}_1, \ldots, \bar{x}_n) \in X \) is called a \textit{Nash equilibrium} for the generalized game \( \mathcal{G} \) if for each \( i \in I \),

\[
\bar{x}_i \in T_i(\bar{x}) \quad \text{and} \quad f_i(\bar{x}_{-i}, \bar{x}_i) \leq f_i(\bar{x}_{-i}, y) \quad \text{for all} \quad y \in T_i(\bar{x}).
\]

In Definition 2.1, if \( T_i(x) := X_i \) for each \( x \in X \) and \( i \in I \), then \( \bar{x}_i \in T_i(\bar{x}) = X_i \) is trivial. In this case, the Nash equilibrium for the generalized game \( \mathcal{G} \) reduces to the Nash equilibrium in [5,8].

We begin with a fixed point theorem which is the basic tool for proving the main result as follow:

**Lemma 2.1.** For each \( i \in I = \{1, 2, \ldots, n\} \), let \( X_i \) be a nonempty compact acyclic finite dimensional ANR, and \( X := \prod_{i=1}^{n} X_i \). For each \( i \in I \), let \( T_i : X \rightarrow 2^{X_i} \) be a closed multimap such that \( T_i(x) \) is a nonempty compact acyclic subset of \( X_i \) for each \( x \in X \), and let \( T : X \rightarrow 2^X \) be a multimap defined by \( T(x) := \prod_{i \in I} T_i(x) \) for each \( x \in X \). Then there exists a fixed point \( \bar{x} \in X \) for \( T \), i.e., for each \( i \in I \), \( \bar{x}_i \in T_i(\bar{x}) \).

**Proof.** Since the product of two acyclic sets is acyclic by the Künneth theorem, each \( T(x) \) is acyclic. Moreover, since a product of a finite family of ANRs is an ANR (e.g., see [2]), \( X = \prod_{i=1}^{n} X_i \) is a compact acyclic finite dimensional ANR. Since each \( T_i \) is a closed multimap with nonempty acyclic values, \( T : X \rightarrow 2^X \) is a closed multimap on \( X \) such that \( T(x) \) is a nonempty compact acyclic subset of a finite dimensional ANR for each \( x \in X \). Furthermore, since an ANR is an lc space (see Proposition 1.2 in [7]), all the hypotheses of Begle’s fixed point theorem [1] are satisfied so that there exists a fixed point \( \bar{x} \in X \) for \( T \), i.e., for each \( i \in I \), \( \bar{x}_i \in T_i(\bar{x}) \). This completes the proof. \( \square \)

**Remarks.** (1) In Lemma 2.1, when \( I \) is singleton and \( X \) is a compact acyclic lc space, Lemma 2.1 reduces to Begle’s fixed point theorem in [1].

(2) Lemma 2.1 is comparable with Theorem 1 in [9] since \( X_i \) need not be convex nor \( X \) is admissible.
The following well-known lemma is a basic tool for the main result of this paper, and for the proof, see [5]:

**Lemma 2.2.** Let $M$ be a complete finite dimensional Riemannian manifold. Then any geodesic convex subset $K$ of $M$ is contractible.

Throughout this paper, all spaces are assumed to be a complete finite dimensional Riemannian manifold, and for some basic definitions and standard terminologies on Riemannian geometry, we shall refer to [5,6], and the references therein.

### 3 Existence of a generalized Nash equilibrium

Recall that when $X$ is a nonempty geodesic convex subset of a finite dimensional Riemannian manifold $M$, then $f : X \to \mathbb{R}$ is called a convex function on $X$ if $f \circ \gamma : [0,1] \to \mathbb{R}$ is convex in the usual sense for every geodesic $\gamma : [0,1] \to X$.

We shall prove the following existence of Nash equilibrium for a generalized game $G = (X_i; T_i, f_i)_{i \in I}$ having compact geodesic convex strategy subsets in finite dimensional Riemannian manifolds:

**Theorem 3.1.** Let $I$ be a finite set of players, and let $G = (X_i; T_i, f_i)_{i \in I}$ be a noncooperative generalized game of normal form, where $X_i$ is a nonempty compact geodesic convex ANR subset of a finite dimensional Riemannian manifold $M_i$ for each player $i \in I$. Assume that the joint strategy space $X := \prod_{i \in I} X_i = X_{-i} \times X_i$ is a subset of $M := \prod_{i \in I} M_i$. Suppose that for each $i \in I$, $f_i : X_{-i} \times X_i \to \mathbb{R}$ is a payoff function and $T_i : X \to 2^{X_i}$ is a closed multimap satisfying the following conditions: For each $i \in I$ and $x = (x_{-i}, x_i) \in X$,

1. $f_i : X_{-i} \times X_i \to \mathbb{R}$ is (jointly) continuous on $X$;
2. $f_i : X_{-i} \times X_i \to \mathbb{R}$ is convex on $X_i$;
3. for each $x \in X$, $T_i(x)$ is nonempty geodesic convex;
4. the function $M_i$, defined on $X$ by

$$M_i(x) := \min_{y \in T_i(x)} f_i(x_{-i}, y)$$

for each $x \in X$,

is upper semicontinuous.

Then there exists an equilibrium point $\bar{x} \in X$ for the game $G$ such that for each $i \in I$,

$$\bar{x}_i \in T_i(\bar{x}) \quad \text{and} \quad f_i(\bar{x}_{-i}, \bar{x}_i) \leq f_i(\bar{x}_{-i}, y) \quad \text{for all} \quad y \in T_i(\bar{x}).$$

**Proof.** First, we note that since a product of a finite family of ANRs is an ANR, $X$ is a finite dimensional ANR. By Lemma 2.2, $X$ is also contractible so
that $X$ is a compact acyclic ANR in a finite dimensional Riemannian manifold. For each $i \in I$, we define a multimap $S_i : X \to 2^{X_i}$ by

$$S_i(x) := \{ y \in T_i(x) \mid f_i(x_{-i}, y) \leq M_i(x) \} \quad \text{for each } x \in X.$$ 

Then, each $S_i(x)$ is nonempty since $T_i(x)$ is compact and $f_i(x_{-i}, \cdot)$ is lower semicontinuous on $T_i(x)$. Next, it is easy to see that $S_i$ has a closed graph in $X \times X_i$. In fact, let $(x_\beta, y_\beta) \in \text{Gr} S_i$ be a net which converges to $(x, y) \in X \times X_i$. Then, $y_\beta \in T_i(x_\beta), (x_\beta) \to x$, and $(y_\beta) \to y$. Since $T_i$ has the closed graph in $X \times X_i$, we have $y \in T_i(x)$. By the assumptions (1) and (4), $f_i$ is (jointly) continuous and $M_i$ is upper semicontinuous so that we have

$$f_i(x_{-i}, y) \leq \liminf_{\beta} f_i(x_{\beta_{-i}}, y_\beta) \leq \liminf_{\beta} M_i(x_\beta) \leq \limsup_{\beta} M_i(x_\beta) \leq M_i(x);$$

which implies that $(x, y) \in \text{Gr} S_i$. Therefore, $S_i$ is a closed multimap such that $S_i(x)$ is a nonempty compact subset of $X_i$ for each $x \in X$ and $i \in I$.

Next, we shall show that for each $i \in I$, $S_i(x)$ is acyclic for each $x \in X$. Since each $S_i(x)$ is nonempty, assume that $y_1, y_2 \in S_i(x) \subseteq T_i(x)$ such that

$$f_i(x_{-i}, y_j) \leq M_i(x) = \min_{y \in T_i(x)} f_i(x_{-i}, y) \quad \text{for each } j = 1, 2.$$ 

Since $T_i(x)$ is geodesic convex, there exists the unique geodesic $\gamma_i : [0, 1] \to X_i$ joining the two points $y_1 = \gamma_i(0), y_2 = \gamma_i(1) \in T_i(x)$ such that $\gamma_i(t) \in T_i(x)$ for every $t \in [0, 1]$. By the convex assumption (2) on $f_i$, for every $t \in [0, 1]$, we have

$$f_i(x_{-i}, \gamma_i(t)) \leq tf_i(x_{-i}, \gamma_i(0)) + (1 - t)f_i(x_{-i}, \gamma_i(1)) = tf_i(x_{-i}, y_1) + (1 - t)f_i(x_{-i}, y_2) \leq M_i(x).$$ 

Consequently, $\gamma_i(t) \in S_i(x)$ for every $t \in [0, 1]$ so that $S_i(x)$ is a nonempty geodesic convex set in the product manifold $M = \Pi_{i \in I} M_i$ endowed with its natural product metric. Then, by Lemma 2.2 again, $S_i(x)$ is acyclic for each $x \in X$. Therefore, all the hypotheses of Lemma 2.1 are satisfied so that there exists a fixed point $\bar{x} \in X$ for $S_i$, i.e., for each $i \in I$, $\bar{x}_i \in S_i(\bar{x})$. Indeed, we obtain an equilibrium point $\bar{x} \in X$ for the game $\mathcal{G}$, i.e., for each $i \in I$,

$$\bar{x}_i \in T_i(\bar{x}) \quad \text{and} \quad f_i(\bar{x}_{-i}, \bar{x}_i) \leq f_i(\bar{x}_{-i}, y) \quad \text{for all } y \in T_i(\bar{x}),$$

which completes the proof. \qed

Remarks. Theorem 3.1 is very different from the previous theorems in [4,5,8,9] in the following aspects:
(a) we do not need the convex assumption on $X_i$;
(b) we do not need the (usual) convex assumption on $f_i$;
(c) we do not need the convex assumption on $T_i(x)$;

In Theorem 3.1, when $T_i(x) := X_i$ for each $i \in I$ and $x \in X$, then the assumptions (3) and (4), and the closedness of $T_i$ are automatically satisfied. Therefore, as a consequence of Theorem 3.1, we can obtain the following existence theorem of a Nash equilibrium due to Kristály [5] as follows:

**Theorem 3.2.** Let $I$ be a finite set of players, and let $\mathcal{G} = (X_i; f_i)_{i \in I}$ be a noncooperative game where $X_i$ is a nonempty compact geodesic convex strategy subset of a finite dimensional Riemannian manifold $M_i$ for each $i \in I$. Assume that for each $i \in I$,

1. $f_i : X_{-i} \times X_i \to \mathbb{R}$ is (jointly) continuous in $X$;
2. $f_i : X_{-i} \times X_i \to \mathbb{R}$ is a convex function on $X_i$.

Then there exists a Nash equilibrium $\bar{x} \in X$ for the game $\mathcal{G}$, i.e., for each $i \in I$,

$$f_j(\bar{x}_{-i}, \bar{x}_i) \leq f_j(\bar{x}_{-i}, y_i) \quad \text{for all } y_i \in X_i$$

By modifying an example in [5], we give an example of a non-convex 2-person game which is suitable for Theorem 3.1, but the previous equilibrium existence theorems due to Ding et al. [4] and Kristály [5] for compact games can not be applied:

**Example 3.3.** Let $\mathcal{G} = (X_i; T_i, f_i)_{i \in I}$ be a non-convex generalized game such that the pure strategic space $X_i$ for each player $i$ is defined by

$$X_1 := \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 \leq x_1, x_2 \leq 1\};$$

$$X_2 := \{(\cos t, \sin t) \in \mathbb{R}^2 \mid \frac{\pi}{4} \leq t \leq \frac{3\pi}{4}\}.$$ 

Then, $X_1$ is a compact convex subset of $\mathbb{R}^2$ in the usual sense, and $X_2$ is compact but not a convex subset of $\mathbb{R}^2$ in the usual sense. However, as remarked in [5], if we consider the Poincaré upper-plane model ($\mathbb{H}^2, g_{\mathbb{H}}$), then the set $X_2$ is geodesic convex with respect to the metric $g_{\mathbb{H}}$ being the image of a geodesic segment from ($\mathbb{H}^2, g_{\mathbb{H}}$).

For each player $i = 1, 2$, let the payoff function $f_i : X = X_1 \times X_2 \to \mathbb{R}$, and a continuous constraint correspondence $T_i : X \to 2^{X_i}$ are defined as follows: for each $((x_1, x_2), (y_1, y_2)) \in X = X_1 \times X_2$,

$$f_1((x_1, x_2), (y_1, y_2)) := x_1 y_1^2 - y_2,$$

$$f_2((x_1, x_2), (y_1, y_2)) := (1 - x_2)(y_1^2 - y_2^2);$$

$$T_1((x_1, x_2), (y_1, y_2)) := X_1, \quad T_2((x_1, x_2), (y_1, y_2)) := X_2.$$
Then the action sets $X_i$ are compact and geodesic convex, and both payoff functions $f_i$ are continuous. It is easy to see that $f_1((\cdot, (y_1, y_2)))$ is convex on $X_1$. Since the function $t \mapsto (1 - x_2) \cos 2t$ is a convex function on $[\frac{\pi}{4}, \frac{3\pi}{4}]$, $f_2((x_1, x_2), \cdot)$ is clearly a convex function on $X_2$.

For each $i \in I$, $T_i$ is continuous on $X$ such that each $T_i(x, y) = X_i$ is a nonempty compact geodesic convex subset of $X_i$. Therefore, we can apply Theorem 3.1 to the generalized game $G = (X_i; T_i, f_i)_{i \in I}$; then we can obtain an equilibrium point $((0, 0), (0, 1)) \in X = X_1 \times X_2$ for $G$ such that

$$(0, 0) \in T_1((0, 0), (0, 1)); \quad (0, 1) \in T_2((0, 0), (0, 1));$$

and

$$-1 = f_1((0, 0), (0, 1)) \leq f_1((x_1, x_2), (0, 1)) \quad \text{for all} \quad (x_1, x_2) \in T_1(x, y) = X_1,$$

$$-1 = f_2((0, 0), (0, 1)) \leq f_2((0, 0), (y_1, y_2)) \quad \text{for all} \quad (y_1, y_2) \in T_2(x, y) = X_2.$$ 

Finally, it should be noted that by following the generalization method from $n$-person game into generalized games with infinite agents or noncompact settings in Ding et al. [4], Theorem 3.1 can be further generalized into noncompact generalized games in finite dimensional Riemannian manifolds.

4 Conclusion

In this paper, using a fixed point theorem for compact acyclic multimaps in a finite dimensional compact acyclic ANR, we prove the existence theorem of Nash equilibrium for the generalized game $G = (X_i; T_i, f_i)_{i \in I}$ of normal form with geodesic convex values, which generalizes the existence theorem of Nash equilibrium due to Kristály in [5].

Acknowledgements. This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology(2012R1A1A2039089).

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Received: March 8, 2016; Published: April 19, 2016