A Note on Properties of the Continuous Weighted OWA Operator

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Abstract

In this paper, we give a correct proof of of Liu [IEEE Trans. Systems, Man, and Cybern. Part B Vol. 36 No. 1 2006 118-127] and Liu [Int. J. Intell. Syst. vol. 20, pp. 1253-1271, 2005]. We also introduce a generalized result of some of which many results of RIM quantifier properties in Liu [19, 20] are corollaries.

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1 Introduction

Since Yager [1] introduced the ordered weighted averaging (OWA) operators [2], they have attracted much interest among researchers. They provide a general class of parameterized aggregation operators that include the min, max, average and have shown themselves to be useful for modelling different kinds of aggregation problems. Many extensions and applications have been proposed [3-10] in the areas of decision making, expert systems, data mining, approximate reasoning fuzzy system and control.

The concept of orness [2] is one of the appealing points in OWA operators. The andlike or orlike aggregation result of an OWA operator is very important both in theory and applications [11-16]. The orness measure reflects the andlike or orlike aggregation result of an OWA operator.
The fuzzy modelling method is somehow arbitrary as the determination of fuzzy rules are mainly based on intuition. The immediate probability can combine these two kind of weights but it does not satisfy the monotonicity for the probability information in uncertain decision making problems [19]. The weighted OWA (WOWA) that can combine these two kind of weights and does not have the limitations of the two other ones.

The WOWA aggregation methods are relatively rare [18, 24], if we compare the researches on the weights obtaining methods in OWA operator, such as the quantifier guided aggregation [2, 21], exponential smoothing [14], especially the maximum entropy method [16, 17, 22, 23]. The most commonly used method is the interpolation function of the weights [18].

Recently, Liu [19, 20] discussed the properties of WOWA operators with respect to orness and proposed an improvement and extension of the OWA operator and RIM quantifier properties in [16] for the WOWA case in Section III of his paper. But the proof of Theorem 4 in [19] and the proof of Theorem 5 in [20] are incomplete. In proving these results, Liu used that \( Q(x) = \int_0^1 f(r)dr \) for some nonnegative function \( f \). But, in general, there does not exist such function if \( Q(x) \) is not absolutely continuous such as Cantor function (see [26]). Therefore, Liu’s proof is incomplete. In this note, we give a correct proof of the theorem. We also introduce a generalized result of some properties of the quantifier generating functions of which many results of RIM quantifier properties in Liu [19, 20] are special cases.

2 Results

An OWA operator of dimension \( n \) is a mapping \( F : \mathbb{R}^n \rightarrow \mathbb{R} \) that has an associated weighting vector \( W = (w_1, w_2, \ldots, w_n) \) having the properties

\[
    w_1 + w_2 + \cdots + w_n = 1; \quad 0 \leq w_j \leq 1 \quad j = 1, 2, \ldots, n
\]

and such that

\[
    F(x_1, x_2, \cdots, x_n) = \sum_{j=1}^{n} w_j y_j
\]

with \( y_j \) being the \( j \)th largest of the \( x_i \). In what follows, we will always assume that \( x_1 \geq x_2 \geq \cdots \geq x_n \), and we will denote this expression as \( F_W(X) \), where \( X = (x_1, x_2, \cdots, x_n) \).

The degree of "orness" associated with this operator is defined as

\[
    \text{orness}(W) = \sum_{j=1}^{n} \frac{n-j}{n-1} w_j
\]
The \( \text{min, max, and average} \) correspond to \( W^*, W_\ast \) and \( W_A \) respectively, where \( W^* = (1, 0, \cdots, 0), W_\ast = (0, 0, \cdots, 1) \) and \( W_A = ((1/n), (1/n), \cdots, (1/n)) \). Obviously, orness\((W^*) = 1 \) and orness\((W_A) = (1/2)\).

In [25], Yager proposed a method for obtaining the OWA weighting vectors via linguistic quantifiers, especially the regular increasing monotone (RIM) quantifier, which can provide information aggregation procedures guided by verbally expressed concepts and a dimension independent description of the desired aggregation.

**Definition 1**. A fuzzy subset \( Q \) of the real line is called a Regular Increasing Monotone (RIM) quantifier if \( Q(0) = 0, Q(1) = 1, \) and \( Q(x) \geq Q(y) \) for \( x > y \).

**Definition 2**. For \( f(x) \) in \([0, 1]\) and a RIM quantifier \( Q(x) \), \( f(x) \) is called generating function of \( Q(x) \), if it satisfies
\[
Q(x) = \int_0^x f(t)dt
\]
where \( f(x) \geq 0, \) and \( \int_0^1 f(x)dx = 1 \).

Obviously, for any differentiable RIM quantifier \( Q(x) \), its generating function \( f(x) \) exists, as \( f(x) \) can be set as its first order differential function \( Q'(x) \) directly.

The quantifier for all and there exist, not none, are defined, respectively, as
\[
\begin{align*}
Q_*(x) &= \begin{cases} 
1 & \text{if } x = 1 \\
0 & \text{if } x \neq 1 
\end{cases} \\
Q^*(x) &= \begin{cases} 
0 & \text{if } x = 0 \\
1 & \text{if } x \neq 0 
\end{cases}
\end{align*}
\]

In [3], Yager extended the orness measure of OWA operator, and defined the orness of a RIM quantifier.

Given a linguistic quantifier \( Q \), we can generate the OWA weights by \( w_i = Q(i/n) - Q((i-1)/n) \). Let \( n \to \infty \), then we can associate with this quantifier a degree of orness as
\[
\text{orness}(Q) = \lim_{n \to \infty} \frac{1}{n-1} \sum_{i=1}^{n-1} \left( Q\left( \frac{i}{n} \right) - Q\left( \frac{i-1}{n} \right) \right)
\]
\[
= \lim_{n \to \infty} \frac{1}{n-1} \sum_{i=1}^{n-1} Q\left( \frac{i}{n} \right)
\]
\[
= \int_0^1 Q(r)dr.
\]
Recently, Liu [19, 20] proposed an improvement and extension of the OWA operator and RIM quantifier properties in [16] for the WOWA case in Section III of his paper. But one of the proof is incomplete. In this note, we give a correct proof of the theorem and give some extension of other result.

**Theorem 1** [19, 20]. For a series of RIM quantifier $Q_k(x)$.
1) If $\lim_{k \to \infty} \text{orness}(Q_k) = 0$, then $\lim_{k \to \infty} Q_k = Q^*$.
2) If $\lim_{k \to \infty} \text{orness}(Q_k) = 1$, then $\lim_{k \to \infty} Q_k = Q^*$.

In proving this result, Liu (Theorem 4 in [19], Theorem 4 in [20]) used that $Q(x) = \int_0^1 f(r)dr$ for some nonnegative function $f$. But, in general, there does not exist such function if $Q(x)$ is not absolutely continuous such as Cantor function (see [26]). Therefore, Liu’s proof is not sufficient. Here, we give a correct proof.

**Proof.** 1) Suppose not, then we have that $\lim_{k \to \infty} Q_k(x) \neq Q^*(x)$.

Then there exists a $x_0 \in (0, 1)$ such that

$$y_0 = \lim_{k \to \infty} Q_k(x_0) > Q^*(x_0) = 0$$

Then there exists a subsequence $\{Q_{k_n}(x_0)\}$ of $\{Q_k(x_0)\}$ such that $\lim_{n \to \infty} Q_{k_n}(x_0) = y_0$. Since $Q_{k_n}$ are non-decreasing function we have,

$$0 = \lim_{k \to \infty} \text{orness}(Q_{k_n}) = \lim_{n \to \infty} \int_0^1 Q_{k_n}(x)dx \geq \lim_{n \to \infty} \int_{x_0}^1 Q_{k_n}(x)dx \geq \frac{y_0}{2}(1-x_0) > 0.$$ 

This is a contradiction.

The proof of 2) is similar.

A property is said to hold *almost everywhere in* $E$ or in abbreviated form, a.e., if it holds in $E$ with Lebesgue measure, say $m$, zero (see [26]).

We generalize Theorem 1 as follows.

**Theorem 2.** If $\lim_{k \to \infty} \int_0^1 Q_k(x)dx = 1 - \alpha$ and $\lim_{k \to \infty} \int_0^1 xQ_k(x)dx = \frac{1-\alpha^2}{2}$, then $\lim_{k \to \infty} Q_k = 1_{[\alpha, 1]}$ a.e.

**Lemma 1.** Let $\{Q_k\}$ be a sequence of RIM. Then there exists a subsequence $\{Q_{k_n}\}$ of $\{Q_k\}$ such that $\lim_{n \to \infty} Q_{k_n}(x) = \lim_{k \to \infty} Q_k(x)$ a.e.

**Proof.** We first show that there exists a subsequence $Q_{k_n}$ of $\{Q_k\}$ such that $\lim_{n \to \infty} Q_{k_n}(x) = \lim Q_k(x)$ for rational number $x$. Let $\{x_i\}, i = 1, 2, 3, \ldots$ be the point of all rational numbers in $(0, 1)$ arranged in a sequence. For $\{x_1\}$, there exists a subsequence, which we shall denote by $\{Q_{1,n}\}$, such that
\{Q_{1,n}(x_1)\} converge to $\lim_{k \to \infty} Q_k(x_1)$ as $n \to \infty$. Let us now consider sequences $S_1, S_2, S_3, \ldots$, which we represent by the array

\begin{align*}
S_1 &: Q_{1,1}, Q_{1,2}, Q_{1,3}, Q_{1,4}, \ldots \\
S_2 &: Q_{2,1}, Q_{2,2}, Q_{2,3}, Q_{2,4}, \ldots \\
S_3 &: Q_{3,1}, Q_{3,2}, Q_{3,3}, Q_{3,4}, \ldots
\end{align*}

and which have the following properties:

(a) $S_n$ is a subsequence of $S_{n-1}$, for $n = 2, 3, 4, \ldots$

(b) $\{Q_{1,n}(x_1)\}$ converges, as $n \to \infty$.

We now go down the diagonal of the array:

\begin{equation*}
S : Q_{1,1} \quad Q_{2,2} \quad Q_{3,3} \quad Q_{4,4} \ldots
\end{equation*}

Then the sequence $S$ is a subsequence of $\{S_n\}$, for $n = 1, 2, 3, \ldots$. Hence (b) implies that $\{Q_{n,n}(x_i)\}$ converges, as $n \to \infty$, for every rational number $x_i$.

Now, noting that $\overline{\lim} Q_k(x)$ is also a RIM, and hence the set of discontinuity points has Lebesgue measure zero. Then the result follows immediately using the denseness of rational numbers.

**Lemma 2.** Let $\{Q(x)\}$ be a RIM satisfying $\int_0^1 Q(x)dx = 1 - \alpha$ and $\int_0^1 xQ(x)dx = \frac{1 - \alpha^2}{2}$, then $Q(x) = 1_{[\alpha,1]}$ a.e.

**Proof.** If $Q(x) = 1_{[\alpha,1]}$ a.e. then clearly we have $\int_0^1 Q(x)dx = 1 - \alpha$ and $\int_0^1 xQ(x)dx = \frac{1 - \alpha^2}{2}$. Let $G(x)$ be another RIM satisfying $\int_0^1 G(x)dx = 1 - \alpha$, $\int_0^1 xG(x)dx = \frac{1 - \alpha^2}{2}$ and suppose that $Q(x) \neq G(x)$, that is, $m\{Q(x) \neq G(x)\} > 0$. Let $Q(x) - G(x) = T(x)$. Since $T \leq 0$ a.e. on $[0, \alpha]$, $T \geq 0$ a.e. on $[\alpha,1]$, and $\int_0^1 T(x)dx = 0$, then $\int_{\alpha}^1 T(x)dx < 0$ and $\int_{\alpha}^1 T(x)dx > 0$. Then we have

\begin{align*}
0 = \int_0^1 xT(x)dx &= \int_0^{\alpha} xT(x)dx + \int_{\alpha}^1 xT(x)dx \\
&> \int_0^{\alpha} \alpha T(x)dx + \int_{\alpha}^1 \alpha T(x)dx \\
&= \alpha \int_0^1 T(x)dx \\
&= 0,
\end{align*}

which is a contradiction.

**Proof of Theorem 2.** By Lemma 1, $\lim_{n \to \infty} Q_{k,n}(x) = \overline{\lim}_{k \to \infty} Q_k(x)$ a.e. By Dominated Convergence Theorem, we have

\begin{equation*}
1 - \alpha = \lim_{k \to \infty} \int_0^1 Q_k(x)dx = \lim_{n \to \infty} \int_0^1 Q_{k,n}(x)dx = \int_0^1 \overline{\lim} Q_k(x)dx
\end{equation*}
and
\[
\frac{1 - \alpha^2}{2} = \lim_{k \to \infty} \int_0^1 xQ_k(x)dx = \lim_{n \to \infty} \int_0^1 xQ_{k_n}(x)dx = \int_0^1 x\lim Q_k(x)dx.
\]

Then we have \(\lim Q_k(x) = 1_{[0,1]}(x)\) a.e. by Lemma 2. Similarly, we have \(\lim Q_k(x) = 1_{[\alpha,1]}\) a.e., and hence we have \(\lim Q_k(x) = 1_{[\alpha,1]}(x)\) a.e.

Liu [19, 20] also proved the following result.

**Theorem 3 [19, 20].** For a series of RIM quantifier \(Q_k(x)\) and for any \(X\).
1) If \(\lim_{k \to \infty} \text{orness}(Q_k) = 0\), then \(\lim_{k \to \infty} F_{P,Q_k}(X) = \min_{1 \leq i \leq n} \{x_i\}\).
2) If \(\lim_{k \to \infty} \text{orness}(Q_k) = 1\), then \(\lim_{k \to \infty} F_{P,Q_k}(X) = \max_{1 \leq i \leq n} \{x_i\}\).

We can generalize above result as follows.

**Theorem 4.** For a sequence of upper semicontinuous RIM quantifier \(Q_k(x)\), for any \(X\) with \(x_1 \geq x_2 \geq \cdots \geq x_n\) and \(0 < \alpha < 1\) with \(\sum_{i=1}^{j} p_i < \alpha < \sum_{i=1}^{j} p_i\), if \(\lim_{k \to \infty} \text{orness}(Q_k) = 1 - \alpha\) and \(\lim_{k \to \infty} \int_0^1 xQ_k(x)dx = \frac{1 - \alpha^2}{2}\), then \(\lim_{k \to \infty} F_{P,Q_k}(x) = \frac{x_j}{x_j}\).

**Proof.** If \(\lim_{k \to \infty} \text{orness}(Q_k) = 1 - \alpha\) and \(\lim_{k \to \infty} \int_0^1 xQ_k(x)dx = \frac{1 - \alpha^2}{2}\), from Theorem 2, we have that \(\lim_{k \to \infty} Q_k(x) = 1_{[\alpha,1]}\) a.e. Since \(\sum_{i=1}^{j} p_i < \alpha < \sum_{i=1}^{j} p_i\), \(\lim_{k \to \infty} \{Q_k(\sum_{i \leq j} p_i) - Q_k(\sum_{i < j} p_i)\} = 1\) for \(l = j\) and \(\lim_{k \to \infty} \{Q_k(\sum_{i \leq j} p_i) - Q_k(\sum_{i < j} p_i)\} = 0\) for \(l \neq j\). So \(\lim_{k \to \infty} F_{P,Q_k}(x) = x_j\), since \(x_1 \geq x_2 \geq \cdots \geq x_n\).

Liu [19, 20] analyzed the following properties of the quantifier generating.

**Theorem 5 [19, 20].** For two RIM quantifiers with generating function \(f(x), g(y)\), if for all \(x,y \in [0, 1], y \leq x, g(x)f(y) \geq f(x)g(y)\), then for all \(t \in [0, 1], Q(t) \geq G(t)\).

**Theorem 6 [20].** For two RIM quantifiers with generating function \(f(x), g(y)\), if for all \(x,y \in [0, 1], y \leq x, f(y) - f(x) \geq g(y) - g(x)\), then for all \(t \in [0, 1], Q(t) \geq G(t)\).

**Theorem 7 [19, 20].** For RIM quantifiers \(Q(x), G(x)\) with differentiable and nonzero generating function \(f(x), g(x)\), if for all \(x \in [0, 1], f(x), g(x) \neq 0\) and \((f'(x)/f(x)) \leq (g'(x)/g(x))\), then for all \(t \in [0, 1], Q(t) \geq G(t)\).

Here we introduce a generalized result of which above results of Liu [19, 20] can be corollaries.

**Theorem 8.** For RIM quantifiers \(Q(x), G(x)\) with generating function \(f(x), g(y)\), if there exists a \(x_0 \in (0, 1)\) such that \(\forall x, x \in [0, x_0], g(x) \leq f(x)\) and \(\forall x, x \in (x_0, 1], g(x) > f(x)\) then for all \(t \in [0, 1], Q(t) \geq G(t)\).
Note on the expected value of a function of a fuzzy variable

Proof. For $t \in [0, x_0]$, we clearly have $Q(t) \geq G(t)$. Now, for $x_0 < t$, since $f(x) - g(x) \geq 0$ for $x \in [0, x_0]$, $g(x) - f(x) \geq 0$ for $x \in (x_0, 1]$, and

$$\int_0^{x_0} (f(x) - g(x)) \, dx = \int_{x_0}^1 (g(x) - f(x)) \, dx,$$

that is, $Q(t) \geq G(t)$.

Now, we can consider Theorem 5, 6, 7 of Liu [19, 20] as corollaries of above result.

Proof of Theorem 5. Since $f$ and $g$ are generating functions, there exists a $x'$ such that $f(x') \geq g(x') > 0$. Then for any $y \leq x'$, $f(x')g(y) \leq g(x')f(y) \leq f(x')f(y)$, and hence $g(y) \leq f(y)$. Now we let $x_0 = \sup \{y : g(y) \leq f(y)\}$, then $\forall x, x' \in [0, x_0], g(x) \leq f(x)$ and $\forall x, x' \in (x_0, 1], g(x) > f(x)$. Therefore, the result follows Theorem 8.

Proof of Theorem 6. Let $T(y) = f(y) - g(y)$ then $T$ is a non-increasing function and $\int_0^1 T(y) \, dy = 0$. Then there exists $x_0$ such that $T(y) = f(y) - g(y) \geq 0$, $y \in [0, x_0]$ and $T(y) = f(y) - g(y) \leq 0$, $y \in (x_0, 1]$, and the functions satisfy the assumption in Theorem 8. Therefore, the result follows Theorem 8.

Proof of Theorem 7. By the assumption, we note that

$$\left( \frac{\log f(x)}{g(x)} \right)' = (\log f(x))' - (\log g(x))' = f'(x)/f(x) - g'(x)/g(x) \leq 0.$$

Then $f(x)/g(x)$ is a non-increasing function and hence there exists $x_0$ such that $f(x)/g(x) \geq 1$, $y \in [0, x_0]$ and $f(x)/g(x) \leq 1$, $y \in (x_0, 1]$, and the functions satisfy the assumption in Theorem 8. Therefore, the result follows Theorem 8.

Assume we have an interval $[a, b]$ and an nonnegative function $p(x)$ defined on $[a, b]$, where $p(x)$ corresponds to the "importance weight" associated with the value $x \in [a, b]$. The OWA aggregation for $([a, b], p)$ can be defined as

$$F_Q([a, b], p) = -\int_a^b x \, dQ(H(x))$$

where $H(x) = \int_x^b p(t) \, dt / \int_a^b p(t) \, dt$.

This is also equivalent to the method of [1, p.1959] with $F_Q([a, b], p) = \int_0^1 Q'(y) H^{-1}(y) \, dy$. In fact, if we let $H(x) = y$, then
\[ F_Q([a, b], p) = -\int_a^b x dQ(H(x)) \]
\[ = \int_0^1 H^{-1}(y) dQ(y) \]
\[ = \int_0^1 H^{-1}(y) Q'(y) dy. \]  \hspace{1cm} (1)

**Theorem 9** [20]. For RIM quantifiers \(Q(x), G(x)\), if for all \(x \in [0, 1], Q(x) \geq G(x)\), for any \(([a, b], p), F_Q([a, b], p) \geq F_G([a, b], p)\).

Here, we consider a theorem of which above results are immediate consequences.

**Theorem 10.** For RIM quantifiers \(Q(x), G(x)\), if for all \(x \in [0, 1], Q(x) \geq G(x)\), then for any non-increasing function \(K\) on \([0, 1]\) such that \(K(0) = 1\) and \(K(1) = 0\),

\[ \int_0^1 K(x) g(x) dx \leq \int_0^1 K(x) f(x) dx. \]

**Proof.** Suppose that \(t \in [0, 1], G(t) \leq Q(t)\). Then we have that

\[ \int_0^1 K(x) g(x) dx = \int_0^1 \int_0^{K(x)} g(x) dx dt \]
\[ = \int_0^1 \int_0^{K^{-1}(t)} g(x) dx dt \]
\[ = \int_0^1 G(K^{-1}(t)) dt \]
\[ \leq \int_0^1 Q(K^{-1}(t)) dt \]
\[ = \int_0^1 K(x) f(x) dx, \]

where \(K^{-1}(t) = \inf \{ x : K(x) \geq t \} \).

**Proof of Theorem 9.** Since \(H(x) = \int_x^b p(t) dt / \int_a^b p(t) dt\) is a non-increasing function, then \(H^{-1}\) is non-increasing. If we regards \(H^{-1}(x)\) in (1) as \(K(x)\) in Theorem 10, we get the desired result.

The following results are immediate from Theorem 8 and 10.

**Corollary 1.** For two RIM quantifiers with generating function \(f(x), g(y)\), if for all \(x, y \in [0, 1], y \leq x, g(x) f(y) \geq f(x) g(y)\), then for any \(([a, b], p), F_Q([a, b], p) \geq F_G([a, b], p)\).
Corollary 2. For two RIM quantifiers with generating function $f(x), g(y)$, if for all $x, y \in [0, 1], y \leq x, f(y) - f(x) \geq g(y) - g(x)$, then for any $([a, b], p), F_Q([a, b], p) \geq F_G([a, b], p)$.

Corollary 3. For RIM quantifiers $Q(x), G(x)$ with differentiable and nonzero generating function $f(x), g(x)$, if for all $x \in [0, 1], f(x), g(x) \neq 0$ and $(f'(x)/f(x)) \leq (g'(x)/g(x))$, then for any $([a, b], p), F_Q([a, b], p) \geq F_G([a, b], p)$.

References


Note on the expected value of a function of a fuzzy variable


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