Dodecagon Quadrangle Systems Having
the Most Large Spectrum

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Abstract

A dodecagon quadrangle is the graph consisting of two cycles: a 12-cycle \((x_1, x_2, ..., x_{12})\) and a 4-cycle \((x_1, x_4, x_7, x_{10})\). A dodecagon quadrangle system \([DQS]\) of order \(v\) and index \(\lambda\) \([DQS_\lambda]\) is a pair \(\Sigma = (X, B)\), where \(X\) is a finite set of \(v\) vertices and \(B\) is a collection of edge disjoint dodecagon quadrangles (called blocks) which partitions the edge set of \(\lambda K_v\), the complete multigraph with vertex set \(X\). In [12] the authors determined the spectrum of \(DQSs\) having index \(\lambda = 1\) and the spectrum of perfect \(DQSs\) in all the cases. In [14] we determine the spectrum of \(DQSs\) of index \(\lambda = 2^n(2h + 1) > 1\), for any \(h \in \mathbb{N}\) and \(n = 0, 1, 2, 3\). In this paper we complete the determination of the spectrum in all the remaining cases.

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1 Introduction

In this paper we complete the research begun in [14], determining completely the spectrum of Dodecagon Quadrangle Systems of any index.

We remember that, in these last years, some design-theorists have studied, \(G\)-decompositions of the complete multigraph \(\lambda K_v\) [1].
In many cases the graph $G$ is a polygon with some chords, forming an inside polygon whose sides joining vertices at distance two. Hexagon triple systems and hexagon triple systems have been studied respectively in [22],[23], while in [2] octagon triple systems are considered. The main problem is generally to determine the spectrum of the corresponding systems and study problems of embedding. For a short historical introduction the reader can see the paper [14], with all the papers cited in the References.

In [12] the authors defined dodecagon quadrangles [DQ-graph], dodecagon quadrangle systems [DQS], perfect dodecagon quadrangle systems [PDQS], determining the spectrum for DQSs having index $\lambda = 1$, and for perfect dodecagon quadrangle systems in all the cases. In these $G$-designs, $G$ is a polygon of 12 vertices with 4 chords which divide it into five 4-cycles. Indeed, a dodecagon quadrangle is the graph formed by a cycle $C_{12} = (x_1, x_2, \ldots, x_{12})$ with the four chords $\{x_1, x_4\}, \{x_4, x_7\}, \{x_7, x_{10}\}, \{x_{10}, x_1\}$. Such a graph it is denoted by $[(x_1), x_2, x_3, (x_4), x_5, x_6, (x_7), x_8, x_9, (x_{10})x_{11}, x_{12}]$ [12][1].

A dodecagon quadrangle system of order $v$ and index $\lambda$ [DQS$_\lambda$] is a pair $\Sigma = (X, B)$, where $X$ is a finite set of $v$ vertices and $B$ is a collection of edge disjoint dodecagon quadrangles (called blocks) which partitions the edge set of $\lambda K_v$, with vertex set $X$ [12][1].

In [12] the spectrum of DQSs having index $\lambda = 1$ has been completely determined, as follows:

**Theorem 1.1** [12]: There exists a DQS of order $v$ and index 1 if and only if: $v \equiv 1 \mod 32, v \geq 33$.

In [14] we determine the spectrum of DQSs of index $\lambda \equiv 2h \mod 4h$, for $h = 1, 2, 4$. This means that we have considered there the cases: $\lambda$ odd, $\lambda = 2(2h + 1)$, $\lambda = 4(2h + 1)$, $\lambda = 8(2h + 1)$, for each $h \in N$, obtaining the following results:

**Theorem 1.2** : There exists a DQS of order $v$ and index $\lambda$ odd if and only if $v \equiv 1 \mod 32, v \geq 33$.

**Theorem 1.3** : There exists a DQS of order $v$ and index $\lambda = 2(2h + 1)$, for any integer $h > 0$, if and only if $v \equiv 1 \mod 16, v \geq 17$.

**Theorem 1.4** : There exists a DQS of order $v$ and index $\lambda = 4(2h + 1)$, for any integer $h > 0$, if and only if $v \equiv 1 \mod 8, v \geq 17$.

**Theorem 1.5** : There exists a DQS of order $v$ and index $\lambda = 8(2h + 1)$, for any integer $h > 0$, if and only if $v \equiv 1 \mod 4, v \geq 13$. 

In this paper we complete the determination of the spectrum in all the remaining cases, which are the DQSs having index \( \lambda = 16(2h+1) \), for any \( h \in \mathbb{N} \). In what follows, we say that a block \( B \) of a DQS \( \Sigma = (X, \mathcal{B}) \) has multiplicity \( r \) if in \( \mathcal{B} \) there are exactly \( r \) blocks \( B_1, B_2, ..., B_r \), such that \( B = B_1 = B_2 = ... = B_r \). In these cases, we will indicate such a situation by \( B^{(r)} \).

2  DQSs of index \( \lambda = 16 \)

The following Theorem is proved in [14].

**Theorem 2.1** : Let \( \Sigma = (X, \mathcal{B}) \) a DQS of order \( v \) and index \( \lambda \).

1. If \( \lambda \) is odd, then \( v \equiv 1 \mod 32 \), \( v \geq 33 \).
2. If \( \lambda \) is even, then for \( \lambda = 2^i(2h+1) \), \( i > 0, h \geq 0, i, h \) integers:
   1. if \( i = 1, 2, 3 \), then \( v \equiv 1 \mod 2^{5-i} \) and \( v \geq 17 \) for \( i = 1, 2, v \geq 13 \) for \( i = 3 \);
   2. if \( i \geq 4 \), then \( v \equiv 1 \mod 2 \), \( v \geq 13 \).

Easily, it follows:

**Theorem 2.2** : Let \( \Sigma = (X, \mathcal{B}) \) a DQS of order \( v \) and index \( \lambda = 16k + 1 \) for any integer \( k > 0 \).

Then, necessarily: \( v \geq 13 \).

**Remark** : Observe that, if for any admissible \( v \) there exists a DQS \( \Sigma \) of order \( v \) and index 16, then we can say that there exists a DQS \( \Sigma' \) having the same order \( v \) of \( \Sigma \) and index 16h, for any \( h \in \mathbb{N} \). Indeed, \( \Sigma' \) can be obtained from \( \Sigma \) by a repetition of blocks.

Therefore, in what follows, we will examine the possible existence of DQSs of index \( \lambda = 16 \) and order \( v \geq 13 \).

3  The cases \( v = 15, 19, 23, 27 \)

**Theorem 3.1** : There exist DQSs of order \( v = 15 \) and index \( \lambda = 16 \).

**Proof**. A possible DQS \( \Sigma = (X, \mathcal{B}) \), of order \( v = 15 \) and index \( \lambda = 16 \), has \( |\mathcal{B}| = 15 \cdot 7 \) blocks. Therefore it can be obtained by difference method, defining 7 base-blocks. The difference-set of \( Z_{15} \) is \( D(15) = \{1, 2, 3, ..., 7\} \). Consider the following DQSs:

\[
B_1 = [(0), 6, 14, (8), 13, 7, (2), 11, 3, (9), 4, 10],
B_{2,3} = [(0), 1, 6, (3), 8, 9, (7), 10, 12, (4), 11, 5]_{(2)},
\]
If $X = Z_{15}$ and $\mathcal{B}$ is the family of all the translates of $B_1, B_{2,3}, B_{4,5}, B_{6,7},$ fixed as base-blocks, then we can verify that $\Sigma = (X, \mathcal{B})$ is a $DQS$ of order $v = 15$ and index $\lambda = 16$. 

**Theorem 3.2**: There exist $DQSs$ of order $v = 19$ and index $\lambda = 16$.

**Proof.** A possible $DQS$ $\Sigma = (X, \mathcal{B})$, of order $v = 19$ and index $\lambda = 16$, has $|\mathcal{B}| = 19 \cdot 9$ blocks. Therefore it can be obtained by difference method, defining 9 base-blocks. The difference-set of $Z_{19}$ is $D(19) = \{1, 2, 3, \ldots, 9\}$. Consider the following $DQS$:

\[
\begin{align*}
B_1 &= [(0), 12, 2, (9), 11, 14, (17), 10, 1, (8), 16, 5], \\
B_2 &= [(0), 13, 3, (9), 11, 14, (17), 10, 1, (8), 16, 5], \\
B_3 &= [(0), 13, 1, (9), 14, 7, (17), 8, 2, (12), 18, 6], \\
B_4 &= [(0), 1, 2, (3), 5, 7, (9), 8, 11, (4), 12, 6]_2, \\
B_5 &= [(0), 6, 10, (5), 8, 11, (1), 12, 9, (7), 3, 2]_2, \\
B_6 &= [(0), 5, 12, (7), 9, 10, (11), 20, 14, (8), 13, 2]_2, \\
B_7 &= [(0), 6, 12, (9), 17, 19, (11), 4, 18, (10), 13, 3]_2, \\
B_8 &= [(0), 17, 21, (5), 9, 10, (11), 8, 3, (6), 13, 12]_2, \\
B_9 &= [(0), 3, 12, (1), 7, 15, (11), 6, 4, (2), 13, 9]_2.
\end{align*}
\]

If $X = Z_{19}$ and $\mathcal{B}$ is the family of all the translates of $B_1, B_2, B_3, B_{4,5}, B_{6,7}, B_{8,9},$ fixed as base-blocks, then we can verify that $\Sigma = (X, \mathcal{B})$ is a $DQS$ of order $v = 19$ and index $\lambda = 16$. 

**Theorem 3.3**: There exist $DQSs$ of order $v = 23$ and index $\lambda = 16$.

**Proof.** A possible $DQS$ $\Sigma = (X, \mathcal{B})$, of order $v = 23$ and index $\lambda = 16$, has $|\mathcal{B}| = 23 \cdot 11$ blocks. Therefore it can be obtained by difference method, defining 11 base-blocks. The difference-set of $Z_{23}$ is $D(23) = \{1, 2, 3, \ldots, 10, 11\}$. Consider the following $DQS$:

\[
\begin{align*}
B_1 &= [(0), 12, 8, (3), 10, 2, (11), 19, 15, (4), 6, 14], \\
B_2 &= [(0), 10, 20, (7), 17, 4, (14), 15, 2, (16), 6, 13], \\
B_3 &= [(0), 11, 22, (10), 14, 18, (13), 1, 12, (4), 16, 15], \\
B_4 &= [(0), 5, 12, (7), 9, 10, (11), 20, 14, (8), 13, 2]_2, \\
B_5 &= [(0), 6, 12, (9), 17, 19, (11), 4, 18, (10), 13, 3]_2, \\
B_6 &= [(0), 17, 21, (5), 9, 10, (11), 8, 3, (6), 13, 12]_2, \\
B_7 &= [(0), 3, 12, (1), 7, 15, (11), 6, 4, (2), 13, 9]_2.
\end{align*}
\]

If $X = Z_{23}$ and $\mathcal{B}$ is the family of all the translates of $B_1, B_2, B_3, B_{4,5}, B_{6,7}, B_{8,9}, B_{10,11},$ fixed as base-blocks, then we can verify that $\Sigma = (X, \mathcal{B})$ is a $DQS$ of order $v = 23$ and index $\lambda = 16$. 

\[
B_{15} = [(0), 4, 3, (5), 12, 8, (7), 9, 10, (6), 2, 13]_2, \\
B_{6,7} = [(0), 2, 7, (5), 1, 4, (11), 8, 9, (3), 6, 14]_2.
\]
Theorem 3.4: There exist DQSs of order $v = 27$ and index $\lambda = 16$.

Proof. A possible DQS $\Sigma = (X, \mathcal{B})$, of order $v = 27$ and index $\lambda = 16$, has $|\mathcal{B}| = 27 \cdot 13$ blocks. Therefore it can be obtained by difference method, defining 13 base-blocks. The difference-set of $Z_{27}$ is $D(27) = \{1, 2, 3, ..., 13\}$. Consider the following DQSs:

$$B_1 = [(0), 10, 23, (9), 25, 11, (2), 19, 3, (13), 7, 14],$$
$$B_2 = [(0), 18, 24, (11), 26, 14, (1), 16, 2, (12), 22, 10],$$
$$B_3 = [(0), 9, 23, (12), 22, 7, (24), 17, 3, (10), 25, 13],$$
$$B_{4,5} = [(0), 3, 12, (1), 7, 15, (11), 6, 4, (2), 13, 9]_2,$$
$$B_{6,7} = [(0), 8, 12, (3), 2, 9, (11), 19, 15, (4), 16, 5]_2,$$
$$B_{8,9} = [(0), 15, 19, (5), 9, 10, (11), 8, 3, (6), 13, 12]_2,$$
$$B_{10,11} = [(0), 5, 12, (7), 9, 10, (11), 20, 14, (8), 13, 2]_2,$$
$$B_{12,13} = [(0), 6, 12, (9), 17, 19, (11), 4, 18, (10), 13, 3]_2.$$

If $X = Z_{27}$ and $\mathcal{B}$ is the family of all the translates of $B_1, B_2, B_3, B_{4,5}, B_{6,7}, B_{8,9}, B_{10,11}, B_{12,13}$, fixed as base-blocks, then we can verify that $\Sigma = (X, \mathcal{B})$ is a DQS of order $v = 27$ and index $\lambda = 16$. \qed

4 Construction $v \to v + 16$

In this Section we give a construction for DQSs.

Theorem 4.1: A DQS of order $v + 16$ and index $\lambda = 16$ can be constructed from a DQS of order $v$ and index $\lambda = 16$.

Proof. Let $\Sigma_1 = (X, \mathcal{B}_1)$ be a DQS of index $\lambda = 16$ and order $v$ odd, $v \geq 13$, and let $\Sigma_2 = (Y, \mathcal{B}_2)$ be a DQS of index $\lambda = 16$ and order $v = 17$, such that $|X \cap Y| = 1$. Further, let

$$X' = \{x_i, y_i, z_i, t_i : i = 1, 2, 3, 4\},$$
$$X = X' \cup \{\infty\},$$
$$Y = Z_{2k} \cup \{\infty\}.$$

Define in $X \cup Z_{2k}$ the family $\mathcal{F}$ of DQSs having for blocks:

$$[(x_1), i + 3, z_3, (i + 4), t_3, i + 5, (y_1), i, x_3, (i + 1), y_3, i + 2]_4,$$
$$[(z_1), i + 3, z_3, (i + 4), t_3, i + 5, (t_1), i, x_3, (i + 1), y_3, i + 2]_4,$$
\[(x_2), i + 3, z_1, (i + 4), t_4, i + 5, (y_2), i, x_4, (i + 1), y_4, i + 2\]_{(4)},

\[(z_2), i + 3, z_1, (i + 4), t_4, i + 5, (t_2), i, x_4, (i + 1), y_4, i + 2\]_{(4)},

for every \(i = 0, 1, 2, ..., 2k\).

If \(B = B_\infty \cup B_\subset \cup F\), it is possible to verify that \(\Sigma = (X \cup Y, B)\) is a DQS of order \(v + 16\) and index \(\lambda = 16\). Therefore, the statement is proved. □

5 Main Results

We have the conclusive results.

**Theorem 5.1**: For every \(v \equiv 1 \mod 4, v \geq 13\), there exists a DQS of order \(v\) and index \(\lambda = 16(2h + 1)\), for any \(h \in N\).

**Proof.** The statement follows from Theorem 1.5, proved in [14], by a repetition of blocks. □

**Theorem 5.2**: For every \(v \equiv 3 \mod 4, v \geq 15\), there exists a DQS of order \(v\) and index \(\lambda = 16\).

**Proof.** The statement follows from Theorems 3.1, 3.2, 3.3, 3.4, for which there exist DQSs having order \(v = 15, 19, 23, 27\) and index \(\lambda = 16\), and from Theorem 4.1, for which, starting from a DQS of index \(\lambda = 16\) and order \(v\), \(v\) odd, \(v \geq 13\), it is possible to construct a DQS of index \(\lambda = 16\) and order \(v + 16\). □

**Theorem 5.3**: For every \(v \equiv 3 \mod 4, v \geq 15\), there exists a DQS of order \(v\) and index \(\lambda = 16(2h + 1)\).

**Proof.** From Theorem 5.2, by a repetition of blocks. □

Collecting together the previous results, we have the following conclusive result:

**Theorem 5.4**: There exist DQS of order \(v\) and index \(\lambda = 16(2h + 1)\) if and only if \(v \equiv 1 \mod 2, v \geq 13\).

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