Intersections among $P^{(3)}(1, 5)$-Designs

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Abstract

A path-hypergraph $P^{(3)}(1, 5)$ is the uniform hypergraph of rank 3, having five vertices and two edges with exactly one vertex in common. In this paper, the intersection problem among $P^{(3)}(1, 5)$-designs of order $v$ is studied, determining the intersection-set $J(v)$ for every admissible $v$.

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1 Introduction

Let $K^{(3)}_v = (X, \mathcal{E})$ be the complete hypergraph, uniform of rank 3, defined in a vertex set $X = \{x_1, x_2, ..., x_v\}$. This means that $\mathcal{E} = \mathcal{P}_3(X)$, the family of all 3-subsets of $X$. Let $H^{(3)}$ be a subhypergraph of $K^{(3)}_v$.

An $H^{(3)}$-design, or an $H^{(3)}$-decomposition of $K^{(3)}_v$, is a pair $\Sigma = (X, \mathcal{B})$, where $\mathcal{B}$ is a partition of the edge-set $\mathcal{P}_3(X)$ of $K^{(3)}_v$ into subsets all of which yields subhypergraphs isomorphic to $H^{(3)}$. The order $v$ of an $H^{(3)}$-design $\Sigma = (X, \mathcal{B})$ is the cardinality of $|X|$. The hypergraphs belonging to $\mathcal{B}$ are called the blocks of $\Sigma$. If $H^{(3)} = K^{(3)}_4$, an $H^{(3)}$-design is a Steiner Quadruple System.

Two $H^{(3)}$-designs $\Sigma_1 = (X, \mathcal{B}_1)$ and $\Sigma_2 = (X, \mathcal{B}_2)$, defined on the same set $X$, are said to have intersection in $k$ blocks provided $|\mathcal{B}_1 \cap \mathcal{B}_2| = k$. If $k = 0$, $\Sigma_1$ and $\Sigma_2$ are said disjoint. The intersection problem for Steiner systems has been studied by C.C.Lindner and A.Rosa for STSs [19], while M.Gionfriddo and C.C.Lindner studied intersections among SQSs [17]. Other
types of intersections had been studied mainly among STSs and SQSs, but also among other G-designs, as it is possible to see in References.

We recall that there exists an $1$-factorization $\mathcal{F} = \{F_1, F_2, ..., F_{v-1}\}$ of the complete graph $K_v$ defined in $X$ (or simply an $1$-factorization of $X$), into the $1$-factors $F_1, F_2, ..., F_{v-1}$, if and only if $v$ is even. In the case $v$ odd, $v = 2k + 1$, there exists a partition $\mathcal{F}^*$ of the edge-set of $K_v$ into $2k + 1$ classes $F_1, F_2, ..., F_v$, such that every class contains $k$ pairwise disjoint edges and, for every $x_i \in X$, $i = 1, 2, ..., v$, the vertex-set of $F_i$ is $X - \{x_i\}$. When $v$ is odd, we call the partition $\mathcal{F}^*$ an $1$-factorization of $K_v$ (of $X$), into the $1$-factors $F_1, F_2, ..., F_v$.

In this paper we study the intersection problem for $P^{(3)}(1, 5)$-designs, where $P^{(3)}(1, 5) = (V, \mathcal{E})$ is the path-hypergraph, having 5 vertices, 2 triples as edges, and exactly one vertex in common between the two triples of $\mathcal{E}$. In other words, if $V = \{x, y_1, y_2, y_3, y_4\}$, then $\mathcal{E} = \{\{x, y_1, y_2\}, \{x, y_3, y_4\}\}$. In what follows, such a hypergraph will be indicated by $[y_1, y_2, (x), y_3, y_4]$ [1]. Further, to be clear, we will call $3$-edges the triples of $\mathcal{E}$ and edges the pairs contained in the triples of $\mathcal{E}$.

Observe that the spectrum of $P^{(3)}(1, 5)$-designs has already known. It has been determined in [3], where it is proved that:

**Theorem 1.1**: There exists a $P^{(3)}(1, 5)$-design of order $v$ if and only if $v \equiv 0, 1, 2 \pmod{4}$, $v \geq 5$.

Following the same symbolism and terminology of [1][5][15][17], if $v$ is an admissible value for the existence of a $P^{(3)}(1, 5)$-design, it will be:

- $p_v = v(v - 1)(v - 2)/12$: number of blocks of any $P^{(3)}(1, 5)$-design;
- $I(v) = \{0, 1, 2, ..., p_v - 2, p_v\}$: set of all the possible values $k$ for which there exist two $P^{(3)}(1, 5)$-designs having exactly $k$ blocks in common;
- $J(v)$: set of all $h \in N$ for which there are two $P^{(3)}(1, 5)$-designs having exactly $k$ blocks in common.

Note that it is always: $p_v - 1 \notin J(v)$, $p_v \in J(v)$, $J(v) \subseteq I(v)$. Further, consider that in [15] the following results are proved:

**Theorem 1.2**: $J(5) = \{1, 5\} = I(5) \setminus \{0, 2, 3\}$, $J(6) = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 10\} = I(6)$, $J(8) = \{0, 1, 2, ..., 26, 28\} = I(8)$.

### 2 Construction

$v = 4h \rightarrow v' = 4h + 4$

In this section we describe a construction $v = 4h \rightarrow v' = 4h + 4$, by which we will study the intersections among $P^{(3)}(1, 5)$-designs having order a multiple of 4.
Construction \( v = 4h \rightarrow v' = 4h + 4 \)

Let \( \Sigma = (X, \mathcal{B}) \) be a \( P^{(3)}(1,5) \)-design of order \( v = 4h \), for any \( h \in N, h \geq 2 \), defined in \( X = \{1, 2, ..., v\} \). Further, let \( Y = \{A, B, C, D\} \) be a set such that \( X \cap Y = \emptyset \). Define the following families of \( P^{(3)}(1,5) \)s.

1) Family \( \Delta \) - Blocks of type \( \Delta \):

\[
\Delta(1) = \{[C, D, (A), 1, B], [A, D, (B), 1, C], [A, B, (C), 1, D], 
[B, C, (D), 1, A], [A, C, (1), B, D]\};
\]

and, for every \( x \in X - \{1\} \):

\[
\Delta(x) = \{[A, B, (x), C, D], [A, C, (x), B, D], [A, D, (x), B, C]\}.
\]

Further: \( \Delta = \bigcup_{x \in X} \Delta(x) \).

2) Family \( \Omega \) - Blocks of type \( \Omega \):

Consider a factorization \( \mathcal{F} = \{F_1, F_2, ..., F_{4h-1}\} \) of \( K_{4h} \) defined in \( X \). Since, for every \( i = 1, 2, ..., 4h - 1 \), every factor \( F_i \in \mathcal{F} \) has cardinality \( 2h \), it is possible to define in each of them a partition \( \Pi_i = \{B_{i,1}, B_{i,2}, ..., B_{i,h-1}, B_{i,h}\} \), in which every class \( B_{i,j} \) has cardinality \( 2 \). Let:

\[
B_{i,1} = \{\{x_{i,1}, y_{i,1}\}, \{z_{i,1}, t_{i,1}\}\},
\]

\[
B_{i,2} = \{\{x_{i,2}, y_{i,2}\}, \{z_{i,2}, t_{i,2}\}\},
\]

............................

\[
B_{i,h} = \{\{x_{i,h}, y_{i,h}\}, \{z_{i,h}, t_{i,h}\}\}.
\]

For every \( y \in Y = \{A, B, C, D\} \) and for every \( i = 1, 2, ..., 4h - 1 \), define the following families of \( P^{(3)}(1,5) \)s.

\[
\Omega(y, i):
\]

\[
[x_{i,1}, y_{i,1}, (y), z_{i,1}, t_{i,1}], [x_{i,2}, y_{i,2}, (y), z_{i,2}, t_{i,2}],
\]

\[
[x_{i,3}, y_{i,3}, (y), z_{i,3}, t_{i,3}], [x_{i,4}, y_{i,4}, (y), z_{i,4}, t_{i,4}],
\]
[x_{i,h-1}, y_{i,h-1}, (y), z_{i,h-1}, t_{i,h-1}], [x_{i,h}, y_{i,h}, (y), z_{i,h}, t_{i,h}].

Further: \( \Omega(y) = \bigcup_{i=1}^{4h-1} \Omega(y, i) \) and \( \Omega = \bigcup_{y \in Y} \Omega(y) \).

If \( X' = X \cup Y \), \( B' = B \cup \Delta \cup \Omega \), we can verify that \( \Sigma' = (X', B') \) is a \( P^{(3)}(1, 5) \)-design of order \( v' = 4h + 4 \).

**Constructions** \( v \to v + 1 \), for \( v \equiv 0 \) or \( 1 \mod 4 \)

Let \( \Sigma = (X, B) \) be a \( P^{(3)}(1, 5) \)-design of order \( v = 4h \geq 8 \) or \( v = 4h + 1 \geq 5 \), defined in \( X = \{1, 2, ..., v\} \). Further, let \( A \) be a vertex not belonging to \( X \). Define in \( X \) a factorization \( F = \{F_1, F_2, ..., F_{4h-1}\} \) of \( K_{4h} \) [respectively \( G = \{G_1, G_2, ..., G_{4h+1}\} \) of \( K_{4h+1} \) in the case \( v = 4h + 1 \)]. Observe that, for every \( i = 1, 2, ..., 4h - 1 \) [resp. \( i = 1, 2, ..., 4h + 1 \)], every factor \( F_i \in F \) \( [G_i \in G] \) has cardinality \( 2h \). Therefore, we can consider a partition \( \Pi_i = \{B_{i,1}, B_{i,2}, ..., B_{i,h-1}, B_{i,h}\} \), in which every class \( B_{i,j} \) has cardinality \( 2 \) and define the families \( \Omega(A, i), \Omega(A), \Omega \), as in the previous Construction \( v \to v + 4 \). If \( X' = X \cup Y \), \( B' = B \cup \Omega \), we can verify that \( \Sigma' = (X', B') \) in a \( P^{(3)}(1, 5) \)-design of order \( v' = 4h + 1 \) [resp. \( 4h + 2 \)].

Observe that the families \( \Delta \) and \( \Omega \), in both Constructions, generate combinatorial structures in \( X \), in which “for every triple of vertices \( x, y, z \) there exists at most one block \( P^{(3)}(1, 5) \) containing them as 3-edge”.

Usually, these structures are called *partial designs*. Therefore, the families \( \Delta, \Omega \) generate *partial \( P^{(3)}(1, 5) \)-designs*. Further, if \( \Sigma' = (V, C') \), \( \Sigma'' = (V, C'') \) are two partial \( P^{(3)}(1, 5) \)-designs, both defined in \( V \), such that a triple \( \{x, y, z\} \subseteq V \) is a 3-edge of a block \( C \in C' \) if and only if \( \{x, y, z\} \) is a 3-edge of a block \( C \in C'' \), then we say that \( \Sigma', \Sigma'' \) are *mutually balanced*.

### 3 \( \Omega \) - Partial \( P^{(3)}(1, 5) \)-designs

In this section we determine all the possible intersections among *mutually balanced* partial \( P^{(3)}(1, 5) \)-designs of type \( \Omega \). We will indicate by:

- \( J_v(\Omega) \) the set of all possible integers \( r \) such that there exist two mutually balanced partial \( P^{(3)}(1, 5) \)-designs of type \( \Omega \), defined on \( v = 4h \) or \( v = 4h + 1 \) vertices, having exactly \( r \) blocks in common;
- and \( J_v(\Omega, y) \) the set of all possible integers \( r \) such that there exist two mutually balanced partial \( P^{(3)}(1, 5) \)-designs of type \( \Omega \), defined on \( v = 4h \) or
v = 4h + 1 vertices, containing in the central position the vertex y and having exactly r blocks in common.

**Theorem 3.1**: \( J_{4h}(\Omega, y) = \{0, 1, 2, ..., 4h^2 - h - 2, 4h^2 - h\} \), for every \( h \geq 2 \).

**Proof.** Let \( h \) be a positive integer, \( h \geq 3 \). Of course, \( 4h^2 - h \in J_{4h}(\Omega, y) \). Fixed any index \( i = 1, 2, ..., 4h - 1 \) and a vertex \( y \in Y = \{A, B, C, D\} \), let \( \Omega(y, i, j) \) be the family of \( P^{(3)}(1, 5)s \), where \( j = 1, 2, ..., h-2, h \), having the blocks:

\[
[x_{i,1}, y_{i,1}, (y), z_{i,1}, t_{i,1}], [x_{i,2}, y_{i,2}, (y), z_{i,2}, t_{i,2}],......
\]
\[
..., [x_{i,j}, y_{i,j}, (y), z_{i,j}, t_{i,j}],
\]
\[
[x_{i,j+1}, y_{i,j+1}, (y), z_{i,j+2}, t_{i,j+2}], [x_{i,j+2}, y_{i,j+2}, (y), z_{i,j+3}, t_{i,j+3}],......
\]
\[
[x_{i,h-1}, y_{i,h-1}, (y), z_{i,h}, t_{i,h}], [x_{i,h}, y_{i,h}, (y), z_{i,j+1}, t_{i,j+1}].
\]

We can observe that the families \( \Omega(y, i) \) and \( \Omega(y, i, j) \) have exactly the same first \( j \) blocks in common (the others are all different). It follows that, for every \( i = 1, 2, ..., 4h - 1 \), it is possible to choose \( j = 1, 2, ..., h - 2, h \) in such a way that:

\[
|\Omega(y, 1) \cap \Omega(y, 1)| = 0, 1, 2, ..., h - 2, h;
\]
\[
|\Omega(y, 2) \cap \Omega(y, 2)| = 0, 1, 2, ..., h - 2, h;
\]
\[
.................................
\]
\[
|\Omega(y, h) \cap \Omega(y, h)| = 0, 1, 2, ..., h - 2, h;
\]

where, for \( h = 2 \), we have only 0,2. Therefore, it is possible to construct two mutually balanced partial \( P^{(3)}(1, 5)s \)-designs of type \( \Omega \), having exactly \( k \) blocks in common, for every \( k = 0, 1, 2, ..., h(4h - 1) - 2, h(4h - 1) \). Hence \( J_{4h}(\Omega, y) = \{0, 1, 2, ..., 4h^2 - h - 2, 4h^2 - h\} \).

**Theorem 3.2**: \( J_{4h+1}(\Omega, y) = \{0, 1, 2, ..., 4h^2 + h - 2, 4h^2 + h\} \) for every \( h \geq 2 \).

**Proof.** Following the same proof of the previous Theorem, we arrive to the same conclusion.
4 $\Delta$ - Partial $P^{(3)}(1,5)$-designs

In this section we determine all the possible intersections among mutually balanced partial $P^{(3)}(1,5)$-designs of type $\Omega$. We will indicate by:

- $J_\nu(\Delta)$ the set of all possible integers $r$ such that there exist two mutually balanced partial $P^{(3)}(1,5)$-designs of type $\Delta$, defined on $v$ vertices, having exactly $r$ blocks in common;
- and $J_\nu(\Delta, x)$ the set of all possible integers $r$ such that there exist two mutually balanced partial $P^{(3)}(1,5)$-designs of type $\Delta$, defined on $v$ vertices, containing in the central position the vertex $x$ and having exactly $r$ blocks in common.

**Theorem 4.1** : $\{0, 2, 4, 6, \ldots, 12h - 6, 12h + 2\} \subseteq J_{4h}(\Delta)$.

**Proof.** Of course, $12h + 2 \in J_{4h}(\Omega)$. Let $x', x'' \in X$. From the Construction $v \rightarrow v + 4$, for $x', x'' \in X - \{1, 2\}$ let

$$\Delta(x', x'') = \Delta(x') \cup \Delta(x'') = \{[A, B, (x'), C, D], [A, C, (x'), B, D], [A, D, (x'), B, C],$$

$$[A, B, (x''), C, D], [A, C, (x''), B, D], [A, D, (x''), B, C]\}.$$  

If:

1) $\Delta_0(x', x'') = \{[x', B, (A), x'', C], [x', C, (D), x'', B], [x', C, (A), x'', D], [x', D, (B), x'', C], [x', D, (A), x'', B], [x', B, (C), x'', D]\};$

2) $\Delta_2(x', x'') = \{[x', B, (A), x'', C], [x', C, (D), x'', B], [x', C, (A), x'', D], [x', D, (B), x'', C], [A, C, (x''), B, D], [A, D, (x''), B, C]\};$

3) $\Delta_3(x', x'') = \{[x', B, (A), x'', C], [x', C, (D), x'', B], [A, D, (x'), B, C],$$

$$[A, B, (x''), C, D], [A, C, (x''), B, D], [A, D, (x''), B, C]\};$

then we can verify that:

$$|\Delta(x', x'') \cap \Delta_0(x', x'')| = 0;$$

$$|\Delta(x', x'') \cap \Delta_2(x', x'')| = 2;$$

$$|\Delta(x', x'') \cap \Delta_3(x', x'')| = 4;$$

further $|\Delta(x', x'') \cap \Delta(x', x'')| = 6.$
If we partition $X - \{1, 2\}$ in $2h - 1$ pairs and exchange in $\Delta(1)$ and $\Delta(2)$ the vertices 1 and 2, we can define two mutually balanced partial $P^{(3)}(1, 5)$-designs, one of them of type $\Delta$, having exactly $k$ blocks in common, for every $k = 0, 2, 4, 6, \ldots, 12h - 6$. \qed

5 Main results

In this section we determine $J(v)$ for $P^{(3)}(1, 5)$-designs.

**Theorem 5.1** : For $P^{(3)}(1, 5)$-designs of order $v = 4h$, $h \geq 2$, it is $J(v) = I(v) = \{0, 1, 2, \ldots, p_v - 2, p_v\}$.

**Proof.** In [15] it is proved that $J(8) = I(8) = \{0, 1, 2, \ldots, 26, 28\}$. Let $\Sigma = (X, \mathcal{B})$ be a $P^{(3)}(1, 5)$-design of order $v = 4h$, $h \geq 2$, and let $\Sigma' = (X', \mathcal{B}')$ be a $P^{(3)}(1, 5)$-design of order $v' = 4h + 4$, obtained by a construction $v \rightarrow v + 4$, described in Section 2. Observe that:

1) $\mathcal{B}' = \mathcal{B} \cup \Delta \cup \Omega$;

2) $\mathcal{B}$, $\Delta$ and $\Omega$ generate in $\Sigma$ three partial $P^{(3)}(1, 5)$-designs, without blocks in common, such that:

$$|\mathcal{B}| = p_v = \frac{4h(4h - 1)(4h - 2)}{3}, \quad |\Delta| = p_\Delta = 12h + 2, \quad |\Omega| = p_\Omega = h(4h - 1);$$

3) if $\mathcal{C}'$, $\mathcal{C}''$ are two mutually balanced partial $P^{(3)}$-designs, defined in the same vertex set contained in $X'$ and $\mathcal{C}' \subseteq \mathcal{B}'$, then $\Sigma'' = (X', \mathcal{C}'')$, where $\mathcal{C}'' = \mathcal{B}' - \mathcal{C}' \cup \mathcal{C}''$, is a $P^{(3)}$-design of order $v'' = 4h + 4$.

Now, use 3) taking $\mathcal{C}'$, $\mathcal{C}''$ both of type $\Delta$, or both of type $\Omega$, or both of type $\Delta$ and $\Omega$ together. By the results of Theorems 3.1, 4.1, we can determine $J(v)$.

Therefore, let $\Sigma_1 = (X, \mathcal{B}_1), \Sigma_2 = (X, \mathcal{B}_2)$ be two $P^{(3)}(1, 5)$-designs of order $v = 4h$, $h \geq 2$, and let $\Sigma'_1 = (X', \mathcal{B}'_1), \Sigma'_2 = (X', \mathcal{B}'_2)$ be two $P^{(3)}(1, 5)$-designs of order $v = 4h + 4$ obtained by Construction $v = 4h \rightarrow v + 4$, with $X' = X \cup \{A, B, C, D\}$, $\mathcal{B}'_1 = \mathcal{B}_1 \cup \Delta_1 \cup \Omega_1$ and $\mathcal{B}'_2 = \mathcal{B}_2 \cup \Delta_2 \cup \Omega_2$.

Further, let $|\mathcal{B}_1 \cap \mathcal{B}_2| = r$, $|\Delta_1 \cap \Delta_2| = s$, $|\Omega_1 \cap \Omega_2| = t$.

Fixed $\Sigma'_1 = (X', \mathcal{B}'_1)$, by Theorems 3.1, 4.1, it is possible to define $\Sigma'_2 = (X', \mathcal{B}'_2)$ such that $r \in J(v)$ and $s = t = 0$. This implies that $J(v) \subseteq J(v + 4)$. Further, it possible to define $\Sigma'_2$ such that $\mathcal{B}_1 = \mathcal{B}_2$, so $r = \frac{h(4h - 1)(4h - 2)}{3}$, $s = 0, 2, 4, \ldots, 12h - 6$ and $t = 0, 1$. It follows that for every $u = 0, 1, 2, 3, \ldots, 12h - 5$

$$\frac{h(4h - 1)(4h - 2)}{3} + u \in J(v + 4).$$
Similarly, for \( r = h(4h - 1)(4h - 2) \), \( s = 0, 2, 4, ..., 12h - 6 \), \( t = 3, 4, ..., 9 \), it follows: \( J(v) \cup J_{4h}(\Delta) \subseteq J(v + 4) \).

To conclude, we observe that it possible to define \( \Sigma'_2 \) such that \( B_1 = B_2 \), \( \Delta_1 = \Delta_2 \), so \( r = h(4h - 1)(4h - 2)/3, s = 12h + 5 \), and \( t \in J_{4h}(\Omega) \). Thus, the statement is proved.

\[ \text{Theorem 5.2:} \quad \text{For } P^{(3)}(1,5)-\text{designs of order } v = 4h + 1, h \geq 2, \text{ it is } J(v) = I(v) = \{0, 1, 2, \ldots, p_v - 2, p_v\}. \]

\[ \text{Proof.} \quad \text{In [15] it is proved that } J(8) = I(8) = \{0, 1, 2, 3, ..., 28, 30\}. \text{ Let } \Sigma = (X, B) \text{ be a } P^{(3)}(1,5)-\text{design of order } v = 4h, h \geq 2, \text{ and let } \Sigma' = (X', B') \text{ be a } P^{(3)}(1,5)-\text{design of order } v' = 4h + 1, \text{ obtained by a construction } v \rightarrow v + 1, \text{ described in Section 2. Following the same technique used in Theorem 5.1, we can arrive the a similar conclusion and prove the statement.} \]

\[ \text{Theorem 5.3:} \quad \text{For } P^{(3)}(1,5)-\text{designs of order } v = 4h + 2, h \geq 1, \text{ it is } J(v) = I(v) = \{0, 1, 2, \ldots, p_v - 2, p_v\}. \]

\[ \text{Proof.} \quad \text{In the Theorem 5.2 it is proved that } J(9) = I(9) = \{0, 1, 2, 3, ..., 40, 42\}. \text{ Let } \Sigma = (X, B) \text{ be a } P^{(3)}(1,5)-\text{design of order } v = 4h + 1, h \geq 2, \text{ and let } \Sigma' = (X', B') \text{ be a } P^{(3)}(1,5)-\text{design of order } v' = 4h + 2, \text{ obtained by a construction } v \rightarrow v + 1, \text{ described in Section 2. Also in this case, following the same technique used in Theorem 5.1, we can prove the statement.} \]

Collecting together Theorems 5.1, 5.2, 5.3, and considering Theorem 1.2 proved in [15], it follows that:

\[ \text{Theorem 5.4:} \quad \text{For } P^{(3)}(1,5)-\text{designs of any admissible order } v \text{ it is } J(v) = I(v) = \{0, 1, 2, \ldots, p_v - 2, p_v\}, \text{ for } v \geq 6 \text{ and } J(5) = \{1, 5\} = I(5) - \{0, 2, 3\}. \]

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\section*{References}


Intersections among ...


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