Existence of Dodecagon Quadrangle Systems

Having Index $\lambda > 1$

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Abstract

A dodecagon quadrangle is the graph consisting of two cycles: a 12-cycle $(x_1, x_2, ..., x_{12})$ and a 4-cycle $(x_1, x_4, x_7, x_{10})$. A dodecagon quadrangle system $[DQS]$ of order $v$ and index $\lambda [DQS]$ is a pair $\Sigma = (X, B)$, where $X$ is a finite set of $v$ vertices and $B$ is a collection of edge disjoint dodecagon quadrangles (called blocks) which partitions the edge set of $\lambda K_v$, the complete multigraph with vertex set $X$. In [11] the authors determined the spectrum of $DQS$s having index $\lambda = 1$ and the spectrum of perfect $DQS$s in all the cases. In this paper we determine the spectrum of $DQS$s of index $\lambda = 2^n(2h + 1) > 1$, for any $h \in \mathbb{N}$ and $n = 0, 1, 2, 3$.

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1 Historical introduction and Definitions

In these last years, $G$-decompositions of the complete multigraph $\lambda K_v$ have been examined mainly in the case in which $G$ is a polygon with some chords, forming an inside polygon whose sides joining vertices at distance two. Hexagon triple systems and hexagon triple systems have been studied respectively in [20],[21], while in [1] octagon triple systems are considered. Generally, in these papers, the authors determine the spectrum of the corresponding systems and
study problems of embedding.

In [2],[3],[5],[14],[15], the authors introduced and studied octagon quadrangle systems, briefly OQSs, perfect OQSs, and OQSs with various types of nestings. They defined an octagon quadrangle as a graph formed by a cycle $C_8(x_1, x_2, ..., x_8)$ with the two chords $\{x_1, x_4\}, \{x_5, x_8\}$.

Balance in $G$-designs have been studied in [10],[4],[13],[22]; while block-colourings have been studied in with [8],[9],[12],[16]. In [11] the authors defined dodecagon quadrangles $[DQ\text{-}graph]$, dodecagon quadrangle systems $[DQS]$, perfect dodecagon quadrangle systems $[PDQS]$, determining the spectrum for $DQS$ s having index $\lambda = 1$, and for perfect dodecagon quadrangle systems in all the cases. In these $G$-designs, $G$ is a polygon of 12 vertices with 4 chords which divide it into five 4-cycles. Indeed, a dodecagon quadrangle is defined as the graph formed by a cycle $C_{12} = (x_1, x_2, ..., x_{12})$ with the four chords $\{x_1, x_4\}, \{x_4, x_7\}, \{x_7, x_{10}\}, \{x_{10}, x_1\}$. Such a graph it is denoted by $[(x_1), x_2, x_3, (x_4), x_5, x_6, (x_7), x_8, x_9, (x_{10}), x_{11}, x_{12}]$ [11]. A dodecagon quadrangle system of order $v$ and index $\lambda [DQS_v]$ is a pair $\Sigma = (X, \mathcal{B})$, where $X$ is a finite set of $v$ vertices and $\mathcal{B}$ is a collection of edge disjoint dodecagon quadrangles (called blocks) which partitions the edge set of $\lambda K_v$, with vertex set $X$ [11].

In [11] the spectrum of $DQS$ s having index $\lambda = 1$ has been completely determined.

**Theorem 1.1** [11]: There exists a $DQS$ of order $v$ and index 1 if and only if: $v \equiv 1 \mod 32, v \geq 33$.

In this paper we determine the spectrum of $DQS$ s of index $\lambda \equiv 2h \mod 4h$, for $h = 1, 2, 4$. This means that we consider the cases: $\lambda$ odd, $\lambda = 2(2h + 1)$, $\lambda = 4(2h + 1)$, $\lambda = 8(2h + 1)$, for each $h \in N$. In what follows, we say that a block $B$ of a $DQS$ $\Sigma = (X, \mathcal{B})$ has multiplicity $r$ if in $\mathcal{B}$ there are exactly $r$ blocks $B_1, B_2, ..., B_r$, such that $B = B_1 = B_2 = ... = B_r$. In these cases, we will indicate such a situation by $B(r)$.

## 2 Necessary existence conditions and existence of $DQS$ s

In this section we prove some necessary existence conditions for $DQS$ s.

**Theorem 2.1** : Let $\Sigma = (X, \mathcal{B})$ a $DQS$ of order $v$ and index $\lambda$.

(1) If $\lambda$ is odd, then $v \equiv 1 \mod 32, v \geq 33$. 

(2) If $\lambda$ is even, then for $\lambda = 2^i(2h + 1)$, $i > 0, h \geq 0$, $i, h$ integers:
(2.1) if $i = 1, 2, 3$, then $v \equiv 1 \mod 2^{5-i}$ and $v \geq 17$ for $i = 1, 2$, $v \geq 13$ for $i = 3$;
(2.2) if $i \geq 4$, then $v \equiv 1 \mod 2$, $v \geq 13$.

**Proof.** Let $\Sigma = (X, \mathcal{B})$ be a DQS of order $v$ and index $\lambda$. Observe that, since all the vertices of a dodecagon quadrangle have degree even, then $v$ must be odd. Further, considering that:

$$b = |\mathcal{B}| = \frac{\lambda v(v - 1)}{32},$$

it follows that:

(1) If $\lambda$ is odd, necessarily: $v \equiv 1 \mod 32$, $v \geq 33$;
(2) If $\lambda$ is even, there exist two integer numbers $i > 0$ and $h \geq 0$ such that $\lambda = 2^i(2h + 1)$. Therefore, if $i = 1, 2, 3$, then $v - 1$ must be a multiple if $2^{5-i}$ and (2.1) holds. If $i \geq 4$, necessarily $v - 1$ is even and $v \geq 13$. $\Box$

### 3 The case $\lambda$ odd

For $\lambda$ odd, the solution follows easily from Theorem 1.1 [11].

**Theorem 3.1:** There exists a DQS of order $v$ and index $\lambda$ odd if and only if $v \equiv 1 \mod 32$, $v \geq 33$.

**Proof.** The necessary existence conditions follow from Theorem 2.1. Therefore, we prove that for every $v \equiv 1 \mod 32$, $v \geq 33$, there exists a DQS of order $v$ and index $\lambda$. Let $\Sigma = (X, \mathcal{B})$ be a DQS of order $v$ and index 1. Further, let $\mathcal{B}'$ be the family of dodecagon quadrangles obtained from $\mathcal{B}$, giving to every block of $\mathcal{B}$ multiplicity $\lambda$. This means that every block of $\mathcal{B}$ is repeated $\lambda$ times in $\mathcal{B}'$. Easily, the system $\Sigma' = (X, \mathcal{B}')$ is a DQS of order $v$ and index $\lambda$ odd. $\Box$

### 4 The case $\lambda \equiv 2, \mod 4$

At first we determine completely the spectrum of DQSs of index $\lambda = 2$. Then, we extend this result to the case $\lambda = 2(2h + 1)$, for any integer $h > 0$.

**Theorem 4.1:** There exists a DQS of order $v$ and index $\lambda = 2$ if and only if $v \equiv 1 \mod 16$, $v \geq 17$. 
Proof. The necessity follows from Theorem 2.1.
If \( v = 17 \) and \( B \) is the following base block defined in \( Z_{17} \):

\[
B = [(0), 4, 8, (5), 13, 12, (10), 16, 9, (2), 3, 6],
\]
then we can verify that \( \Sigma = (Z_{17}, B) \), where \( B \) is the collection of all the translates of \( B \), is a DQS of order \( v = 17 \) and index \( \lambda = 2 \).

Let \( v \) be a positive integer such that \( v \equiv 1 \mod 16 \), \( v > 17 \).
Now, let \( (x, i) = x_i \), \( Z_{8,i} = Z_8 \times \{i\} \), \( i = 1, 2, ..., 2k \), \( Z_{8,j} = Z_8 \times \{j\} \), \( j = A, B \), \( \infty \notin Z_{8,i} \cup Z_{8,j} \), for every \( i, j \).

Let \( \Sigma_1 = (X_1, B_1) \) be a DQS of order \( v' = 16k + 1 \), \( v' \geq 17 \), and index \( \lambda' = 2 \), and let \( \Sigma_2 = (X_2, B_2) \) be a DQS of order \( v'' = 17 \) and index \( \lambda'' = 2 \), where:

\[
X_1 = \bigcup_{i=1}^{2k} Z_{8,i} \cup \{\infty\}, \text{ and}
X_2 = Z_{8,A} \cup Z_{8,B} \cup \{\infty\}.
\]

For every \( i = 1, 2, ..., 2k \) and for every \( j = A, B \), consider, at first, the following dodecagon quadrangles all defined in \( Z_{8,i} \cup Z_{8,j} \):

\[
F_{i,j,1} = [(1)_i, 5_j, 5_i, (1)_j, 6_i, 7_j, (2)_i, 0_j, 0_i, (2)_j, 7_i, 6_j],
\]
\[
F_{i,j,2} = [(1)_i, 7_j, 5_i, (3)_j, 6_i, 5_j, (2)_i, 6_j, 0_i, (4)_j, 7_i, 0_j],
\]
\[
F_{i,j,3} = [(3)_i, 6_j, 6_i, (2)_j, 5_i, 0_j, (4)_i, 7_j, 7_i, (1)_j, 0_i, 5_j],
\]
\[
F_{i,j,4} = [(3)_i, 0_j, 6_i, (4)_j, 5_i, 6_j, (4)_i, 5_j, 7_i, (3)_j, 0_i, 7_j].
\]

Then, indicate by \( \mathcal{F} \) the family of all the blocks \( (F_{i,j,1})_{(2)}, (F_{i,j,2})_{(2)}, (F_{i,j,3})_{(2)}, (F_{i,j,4})_{(2)} \), where every block \( F_{i,j,u} \), for \( u = 1, 2, 3, 4 \), is considered 2-times.

If \( X = X_1 \cup X_2 \) and \( \mathcal{D} = B_1 \cup B_2 \cup \mathcal{F} \), then we can verify that \( \Sigma = (X, \mathcal{D}) \) is a DQS of order \( v = v' + v'' - 1 = 16k + 17 \) and index \( \lambda = 2 \).
Indeed, we can see that:

1) all the pairs \( x, y \) belonging to \( X_1 \) are contained, as edge, two times in the blocks of \( B_1 \);
2) all the pairs \( x, y \) belonging to \( X_2 \) are contained, as edge, two times in the blocks of \( B_2 \);
3) all the pairs \(x, y\), such that \(x \in X_1 - \{\infty\}\) and \(y \in X_2 - \{\infty\}\), are contained, as edge, two times in the blocks of \(\mathcal{F}\).

Therefore, the case \(\lambda = 2\) is completely solved.

**Theorem 4.2**: There exists a DQS of order \(v\) and index \(\lambda = 2(2h + 1)\), for any integer \(h > 0\), if and only if \(v \equiv 1 \mod 16\), \(v \geq 17\).

**Proof.** The statement follows easily from Theorem 2.1, by repetition of blocks. For every DQS or order \(v\) and index \(\lambda = 2\), it is possible to define a DQS of order \(v\) and index \(\lambda' = 2(2h + 1)\), for any integer \(h > 0\), repeating every block \(2h + 1\) times. \(\square\)

5 The case \(\lambda \equiv 4, \mod 8\)

Also in this section, at first we determine completely the spectrum of DQSs of index \(\lambda = 4\), then we extend the result to the case \(\lambda = 2(2h + 1)\), for any integer \(h > 0\).

**Theorem 5.1**: There exists a DQS of order \(v\) and index \(\lambda = 4\) if and only if \(v \equiv 1 \mod 8\), \(v \geq 13\).

**Proof.** The necessity follows from Theorem 2.1. Therefore, consider a positive integer \(v\) such that \(v \equiv 1 \mod 8\), \(v \geq 17\). The existence for \(v = 17\) follows from Theorem 3.1, by a repetition of blocks, giving to each of them multiplicity 2.

Therefore, let \(v = 25\) and let \(B_1, B_2, B_3\) be the following base blocks defined in \(\mathbb{Z}_{25}\):

\[
B_1 = [(0), 22, 7, (1), 24, 14, (13), 17, 8, (2), 10, 9],
\]

\[
B_2 = [(0), 11, 16, (5), 17, 19, (13), 23, 22, (6), 9, 20],
\]

\[
B_3 = [(0), 12, 20, (3), 10, 17, (13), 1, 8, (4), 7, 2].
\]

Similarly to the previous Theorem, let \(\Sigma_1 = (X_1, B_1)\) be a DQS of order \(v' = 16k + 1\) or \(v' = 16k + 9, v' \geq 17\), and index \(\lambda' = 4\), and let \(\Sigma_2 = (X_2, B_2)\) be a DQS of order \(v'' = 17\) and index \(\lambda'' = 4\). By the same construction of Theorem 3.1, we can define the blocks \(F_{i,j,1}, F_{i,j,2}, F_{i,j,3}, F_{i,j,4}\) and the family \(\mathcal{F}\) containing all the blocks \(F_{i,j,1}^{(4)}, F_{i,j,2}^{(4)}, F_{i,j,3}^{(4)}, F_{i,j,4}^{(4)}\) having multiplicity 4.

If \(X = X_1 \cup X_2\) and \(\mathcal{D} = B_1 \cup B_2 \cup \mathcal{F}\), then we can verify that \(\Sigma = (X, \mathcal{D})\) is a DQS of order \(v = 16k + 17\) or \(v = 16k + 25\), and index \(\lambda = 4\). \(\square\)
Theorem 5.2: There exists a DQS of order \( v \) and index \( \lambda = 4(2h+1) \), for any integer \( h > 0 \), if and only if \( v \equiv 1 \mod 8 \), \( v \geq 17 \).

Proof. The statement follows from Theorem 3.1, by repetition of blocks. For every DQS of order \( v \) and index \( \lambda = 4 \), it is possible to define a DQS of order \( v \) and index \( \lambda' = 4(2h+1) \), for any integer \( h > 0 \), repeating every block \( 2h+1 \) times. \( \square \)

6 The case \( \lambda \equiv 8 \mod 16 \)

In this section we determine completely the spectrum of DQSs having index \( \lambda = 8(2h+1) \), \( h \in \mathbb{N} \).

Theorem 6.1: There exists a DQS of order \( v \) and index \( \lambda = 8 \) if and only if \( v \equiv 1 \mod 4 \), \( v \geq 13 \).

Proof. The necessity follows from Theorem 2.1. Therefore, consider a positive integer \( v \) such that \( v \equiv 1 \mod 4 \), \( v \geq 13 \).

The existence for \( v = 17 \) follows from Theorem 3.1, by a repetition of blocks, giving to each of them multiplicity 4.

1) Let \( v = 13 \). Define in \( X = \mathbb{Z}_{13} \) the following base-blocks:

\[
B_1 = [(0), 1, 2, (3), 5, 7, (9), 8, 11, (4), 12, 6], \\
B_2 = [(0), 6, 10, (5), 8, 11, (1), 12, 9, (7), 3, 2], \\
B_3 = [(0), 8, 12, (2), 7, 11, (3), 5, 6, (10), 9, 4].
\]

If \( B \) is the collection of all the translates of \( B_1, B_2, B_3 \), then \( \Sigma = (X, B) \) is a DQS of order \( v = 13 \) and index \( \lambda = 8 \).

2) Let \( v = 21 \). Define in \( X = \mathbb{Z}_{21} \) the following base-blocks:

\[
B_1 = [(0), 3, 12, (1), 7, 15, (11), 6, 4, (2), 13, 9], \\
B_2 = [(0), 8, 12, (3), 2, 9, (11), 19, 15(4), 16, 5], \\
B_3 = [(0), 15, 19, (5), 9, 10, (11), 8, 3, (6), 13, 12], \\
B_4 = [(0), 5, 12, (7), 9, 10, (11), 20, 14, (8), 13, 2], \\
B_5 = [(0), 6, 12, (9), 17, 19, (11), 4, 18, (10), 13, 3].
\]

If \( B \) is the collection of all the translates of \( B_1, B_2, B_3, B_4, B_5 \), then \( \Sigma = (X, B) \) is a DQS of order \( v = 21 \) and index \( \lambda = 8 \).
3) Construction $v \rightarrow v + 12$.

Similarly to the previous Theorem, let $\Sigma_1 = (X_1, B_1)$ be a $DQS(v')$ of order $v' = 16k + 1$ or $v' = 16k + 9$, $v' \geq 17$, and index $\lambda' = 4$, and let $\Sigma_2 = (X_2, B_2)$ be a $DQS(v'')$ of order $v'' = 17$ and index $\lambda'' = 4$. By the same construction of Theorem 3.1, we can define the blocks $F_{i,j,1}', F_{i,j,2}', F_{i,j,3}', F_{i,j,4}'$ and the family $\mathcal{F}$ containing all the blocks $F_{i,j,1}'', F_{i,j,2}'', F_{i,j,3}'', F_{i,j,4}''$ having multiplicity 4.

If $X = X_1 \cup X_2$ and $\mathcal{D} = B_1 \cup B_2 \cup \mathcal{F}$, then we can verify that $\Sigma = (X, \mathcal{D})$ is a $DQS$ of order $v = 16k + 17$ or $v = 16k + 25$, and index $\lambda = 4$. □

**Theorem 6.2** : There exists a $DQS$ of order $v$ and index $\lambda = 8(2h + 1)$, for any integer $h > 0$, if and only if $v \equiv 1 \mod 4$, $v \geq 13$.

**Proof.** Similarly to the other cases, the statement follows from Theorem 3.1, by repetition of blocks. For every $DQS$ or order $v$ and index $\lambda = 8$, it is possible to define a $DQS$ of order $v$ and index $\lambda' = 8(2h + 1)$, for any integer $h > 0$, repeating every block $2h + 1$ times. □

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**References**


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