One-Step Hybrid Block Method with One Generalized Off-Step Points for Direct Solution of Second Order Ordinary Differential Equations

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Abstract

Our focus in this paper is the development, analysis and implementation of a new hybrid block method with one generalized off step point for solving second order ordinary differential equation directly. In the derivation of the method, power series is adopted as basis function to obtain the main continuous scheme through collocation and interpolations approach. Taylor method is also used together with new method to generate the non-overlapping numerical results. As required by all numerical methods, the numerical properties of the new block which include convergent and stability region are also investigated. The developed method was found to compare favourably with the existing methods in term of error.

Mathematics Subject Classification: 65L05, 65L06, 65L20
Keywords: Hybrid method, Block method, Second order differential equation, Power series, One off step points

1 Introduction

This article considered the solution to the general second order initial value problem (IVPs) of the form

$$y'' = f(x, y, y'), \quad y(a) = \eta_0, \quad y'(a) = \eta_1, \quad x \in [a, b].$$

(1)

Class of Equations (1) often arises in numerous areas of engineering and sciences such as physics, biology and chemical. Mostly, these equations have not analytical solution. For that reason, numerical methods become very crucial. Indeed, several numerical methods for solving Equation (1) have been proposed, for example, Euler method, Runge-Kutta, linear multistep method, predictor - corrector method, block method and hybrid method (see [4], [7] and [8]). However, these methods have their drawbacks which effects on their accuracy and efficiency. Recently, hybrid block methods for solving equations (1) directly have been proposed. In the latter, the researchers have tried to combine advantages of direct, block and hybrid methods (see [1], [2], [3], [5] and [6]) to overcome the zero stability problem in linear multistep as well as to avoid setbacks in reduction methods and generating numerical results concurrently.

2 Methodology

In this section, one step block hybrid method with one generalized off step points i.e $x_{n+s}$ for solving (1) is derived.

Let the approximate solution of (1) to be the power series polynomial of the form:

$$y(x) = \sum_{i=0}^{v+m-1} a_i \left( \frac{x - x_n}{h} \right)^i. \quad (2)$$

where,

i- $x \in [x_n, x_{n+1}]$ for $n = 0, 1, 2, ..., N - 1$,

ii- $v$ denotes of the number of interpolation points which is equal to the order of differential equation,

iii- $m$ represents the number of collocation points,

iv- $h = x_n - x_{n-1}$ is constant step size of partition of interval $[a, b]$ which is given by $a = x_0 < x_1 < ... < x_{N-1} < x_N = b$. 
Differentiating (2) twice gives

\[ y''(x) = f(x, y, y') = \sum_{i=2}^{v+m-1} \frac{i(i-1)}{h^2} a_i \left( \frac{x-x_n}{h} \right)^{i-2}. \]  \hspace{1cm} (3)

Interpolating (2) at \( x_n, x_{n+s} \) and collocating (3) at all points in the selected interval produces five equations which can be written in matrix of the form:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & s & s^2 & s^3 & s^4 \\
0 & 0 & \frac{s}{h^2} & 0 & 0 \\
0 & 0 & \frac{2s}{h^2} & \frac{6s}{h^2} & \frac{12s^2}{h^2} \\
0 & 0 & \frac{2s}{h^2} & \frac{6s}{h^2} & \frac{12s^2}{h^2}
\end{pmatrix}
\begin{pmatrix}
a_0 \\
a_1 \\
a_2 \\
a_3 \\
a_4
\end{pmatrix}
= \begin{pmatrix}
y_n \\
y_{n+s} \\
f_n \\
f_{n+s} \\
 f_{n+1}
\end{pmatrix}. \hspace{1cm} (4)

Applying Gaussian elimination method to (5) gives

\[ a_0 = y_n \]
\[ a_1 = \frac{(12s - 12)}{12s(s - 1)} y_n + \frac{(12s - 12)}{12s(s - 1)} y_{n+s} + \frac{(h^2 s^4 - 5h^2 s^3 + 4h^2 s^2)}{12s(s - 1)} f_n \]
\[ + \frac{(2h^2 s^2 - h^2 s^3)}{12s(s - 1)} f_{n+s} - \frac{(h^2 s^3)}{12(s - 1)} f_{n+1} \]
\[ a_2 = \frac{(f_n h^2)}{2} \]
\[ a_3 = \frac{h^2 s}{6(s-1)} f_{n+1} - \frac{h^2 (s^2 - 1)}{6s(s-1)} f_n - \frac{h^2}{6s(s-1)} f_{n+s} \]
\[ a_4 = \frac{-h^2}{12(s-1)} f_{n+1} + \frac{h^2}{12s} f_n + \frac{h^2}{12s(s-1)} f_{n+s} \] \hspace{1cm} (5)

The values of \( a_i', i = 0(1)5 \) are substituted back into equation (2) to give a continuous implicit scheme of the form

\[ y(x) = \sum_{i=0,s} \alpha_i(x) y_{n+i} + \sum_{i=0,s,1} \beta_i(x) f_{n+i} \] \hspace{1cm} (6)

The first derivative of equation (6) gives

\[ y'(x) = \sum_{i=0,s} \frac{\partial}{\partial x} \alpha_i(x) y_{n+i} + \sum_{i=0,s,1} \frac{\partial}{\partial x} \beta_i(x) f_{n+i} \] \hspace{1cm} (7)

where

\[ \alpha_0 = \frac{1 - (x - x_n)}{h s} \]
\[ \alpha_s = \frac{x - x_n}{h s} \]
\[ \beta_0 = \frac{(x - x_n)(x_n - x + h s)}{12h^2 s} \left( h^2 s^2 - 4h^2 s + h s x - h s x_n + 2 h x - 2 h x_n \right) \]
\[ - x^2 + 2 x x_n - x_n^2 \]
\[ \beta_s = \frac{(x_n - x)(x_n - x + hs)}{12h^2s(s - 1)} \left( h^2s^2 - 2h^2s + hsx - hsx_n - 2hx + 2hx_n \right) + x^2 - 2xx_n + x_n^2 \]
\[ \beta_1 = \frac{-(x - x_n)(x_n - x + hs)(h^2s^2 + hsx - hsx_n - x^2 + 2xx_n - x_n^2)}{12h^2(s - 1)} \]

Evaluating equation (6) at the non-interpolating point \( x_{n+1} \), this yields
\[
y_{n+1} - \frac{y_{n+s}}{s} = \frac{(12s - 12)}{12s} y_n + \frac{(h^2s^3 - 4h^2s^2 + 4h^2s - h^2)}{12s} f_n
+ \frac{(-h^2s^2 + h^2s + h^2)}{12s} f_{n+s} - \frac{(h^2s^3 + h^2s^2 - h^2s)}{12s} f_{n+1} \tag{8} \]

Equation (7) is evaluated at all points \( x_n, x_{n+s}, x_{n+1} \) to produce
\[
y'_{n+1} - \frac{(12s - 12)}{12hs(s - 1)} y_{n+s} = -\frac{(12s - 12)}{12hs(s - 1)} y_n + \frac{(h^2s^4 - 5h^2s^3 + 4h^2s^2)}{12hs(s - 1)} f_n
+ \frac{2h^2s^2 - h^2s^3}{12hs(s - 1)} f_{n+s} - \frac{h^2s^3}{12(s - 1)} f_{n+1} \tag{9} \]
\[
y_{n+s} - \frac{(12s - 12)}{12hs(s - 1)} y_{n+s} = -(12s - 12) \frac{y_n}{12hs(s - 1)} + \frac{(h^2s^4 - 3h^2s^3 + 2h^2s^2)}{12hs(s - 1)} f_n
- \frac{(4h^2s^3 - 3h^2s^2)}{12hs(s - 1)} f_{n+s} + \frac{h^2s^3}{12(s - 1)} f_{n+1} \tag{10} \]
\[
y'_{n+1} - \frac{(12s - 12)}{12hs(s - 1)} y_{n+s} = \frac{h^2s^4 - 5h^2s^3 + 10h^2s^2 - 8h^2s + 2h^2}{12hs(s - 1)} f_n
- \frac{12(s - 1)}{12hs(s - 1)} y_n - \frac{(h^2s^4 - 2h^2s^2 + 2h^2)}{12hs(s - 1)} f_{n+s} - \frac{h^2s^4 - 6h^2s^2 + 4h^2s}{12hs(s - 1)} f_{n+1} \tag{11} \]

Combining equation (8)-(11) will produce a block of the form:
\[
A^{[1]2}Y^{[1]2}_m = B^{[1]2}_1 R^{[1]2}_1 + h^2 \left[ D^{[1]2} R^{[1]2}_2 + E^{[1]2} R^{[1]2}_3 \right] \tag{12} 
\]

\[
A^{[1]2} = \begin{pmatrix}
\frac{-1}{s} & 1 & 0 & 0 \\
\text替代} & 0 & 0 & 0 \\
\text替代} & 0 & 1 & 0 \\
\text替代} & 0 & 0 & 1 \\
\end{pmatrix}, \quad Y^{[1]2}_m = \begin{pmatrix}
y_{n+s} \\
y_{n+1} \\
y_{n+s} \\
y_{n+1} \\
\end{pmatrix}, \quad B^{[1]2}_1 = \begin{pmatrix}
0 & \frac{s-1}{s} & 0 & 0 \\
0 & \frac{1}{hs} & 0 & -1 \\
0 & \frac{h^2}{hs} & 0 & 0 \\
0 & \frac{0}{h^2} & 0 & 0 \\
\end{pmatrix}, \quad B^{[1]2}_2 = \begin{pmatrix}
0 & 0 & 0 & (s-1)(s^2-3s+1) \\
0 & 0 & 0 & \frac{(s^2-s-4)}{(12s)} \\
0 & 0 & 0 & \frac{(s^2-s-4)}{(12s)} \\
0 & 0 & 0 & \frac{(s^2-4s+6s-2)}{(12s)} \\
\end{pmatrix}, \quad B^{[1]2}_3 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix} 
\]

\[
R^{[1]2}_1 = \begin{pmatrix}
y_{n-1} \\
y_n \\
y'_{n-1} \\
y'_{n} \\
\end{pmatrix}, \quad D^{[1]2} = \begin{pmatrix}
0 & 0 & 0 & \frac{(s-1)(s^2-3s+1)}{(12s)} \\
0 & 0 & 0 & \frac{(s^2-s-4)}{(12s)} \\
0 & 0 & 0 & \frac{(s^2-s-4)}{(12s)} \\
0 & 0 & 0 & \frac{(s^2-4s+6s-2)}{(12s)} \\
\end{pmatrix}, \quad R^{[1]2}_2 = \begin{pmatrix}
0 & 0 & 0 & f_{n-1} \\
0 & 0 & 0 & f_{n-2} \\
0 & 0 & 0 & f_{n-3} \\
0 & 0 & 0 & f_n \\
\end{pmatrix} 
\]
Multiplying Equation (12) by the inverse of $A^{[1]2}$ gives

$$IY_m = \bar{B}^{[1]2}R_1^{[1]2} + h^2 \left[ \bar{D}^{[1]2}R_2^{[1]2} + \bar{E}^{[1]2}R_3^{[1]2} \right]$$

where $I$ is $4 \times 4$ identity matrix and

$$E^{[1]2} = \begin{pmatrix}
0 & 0 & \frac{s^2(s-4)}{12(s-1)} & \frac{s^2(s-4)}{12(s-1)} \\
0 & 0 & \frac{12s(s-1)}{s(s-3)} & \frac{12s(s-1)}{s(s-3)} \\
0 & 0 & \frac{6(s-1)}{6s(s-1)} & \frac{6(s-1)}{6s(s-1)} \\
0 & 0 & \frac{1}{6s(s-1)} & \frac{1}{6s(s-1)}
\end{pmatrix}$$

which can be written as

$$y_{n+s} = y_n + \frac{h^2s^2(s-4)}{12}f_n + \frac{h^2s^2(s-2)}{12(s-12)}f_{n+s} + \frac{h^2s^4}{12s-12}f_{n+1}$$

$$y_{n+1} = y_n + hs'y_n - \frac{h^2(4s-1)}{12s}f_n - \frac{h^2}{12s(s-1)}f_{n+s} + \frac{h^2(2s-1)}{12(s-1)}f_{n+1}$$

$$y'_{n+s} = y_n - \frac{hs(s-3)}{6}f_n + \frac{hs(2s-3)}{6(s-1)}f_{n+s} + \frac{hs^3}{6(s-1)}f_{n+1}$$

$$y'_{n+1} = y_n + \frac{h(3s-1)}{6s}f_n + \frac{-h}{6(s-1)}f_{n+s} + \frac{h(3s-2)}{6(s-1)}f_{n+1}$$

### 3 Analysis of the Method

#### 3.1 Order of the Method

The linear difference operator $L$ associated with (13) is defined as

$$L[y(x); h] = IY_m - \bar{B}^{[1]2}R_1^{[1]2} - h^2 \left[ \bar{D}^{[1]2}R_2^{[1]2} + \bar{E}^{[1]2}R_3^{[1]2} \right]$$
where \( y(x) \) is an arbitrary test function continuously differentiable on \([a,b]\). Y\(_m\) and \( R_3^{[1]}\) components are expanded in Taylors series respectively and its terms are collected in powers of \( h \) to give

\[
L[y(x), h] = C_0 y(x) + C_1 h y'(x) + C_2 h y''(x) + \cdots \tag{19}
\]

**Definition 3.1** Hybrid block method (13) and associated linear operator (18) are said to be of order \( p \), if \( C_0 = \bar{C}_1 = \cdots = \bar{C}_{p+2} = 0 \) and \( \bar{C}_{p+2} \neq 0 \) with error vector constants \( \bar{C}_{p+2} \).

Expanding (13) in Taylor series about \( x \) gives

\[
\begin{bmatrix}
\sum_{j=0}^{\infty} \frac{s^j h^j}{j!} y_n - y_n - (sh) y_n' + \frac{s^2 (s-4) h^2}{12} y_n'' - \frac{s^4 h^4}{128 - 12} \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} y_n^{j+2} \\
\sum_{j=0}^{\infty} \frac{h^j}{j!} y_n' - y_n - h y_n' + \frac{(4s-1) h^2}{128 - 12} y_n'' - \frac{2s^2 - 1}{128 - 12} \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} y_n^{j+2} \\
\sum_{j=0}^{\infty} \frac{s^j h^j}{j!} y_n - h y_n' + \frac{s (s-3) h}{6} y_n'' - \frac{s (2s-3)}{6(s-1)} \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y_n^{j+2} \\
\sum_{j=0}^{\infty} \frac{h^j}{j!} y_n' + \frac{1}{6s (s-1)} \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y_n^{j+2} \\
\end{bmatrix}
= 0
\]

By comparing the coefficient of \( h \), we obtain the order of the method to be 
\([3, 3, 3, 3]^T\) with error constant

\[
\bar{C}_5 = \begin{bmatrix}
120 s^4 - 48 s^3 \\
8640 s^4 - 120 s^4 \\
24 s^3 - 4 s^4 \\
0
\end{bmatrix}
\]

which is true for all \( s \in (0, 1) \setminus \{ s = \frac{48}{120} , \frac{1}{2} \} \)

### 3.2 Zero Stability

The hybrid block method (13) is said to be zero stable if the first characteristic polynomial \( \pi(r) \) having roots such that \(|r_z| > 1\), and if \(|r_z| = 1\), then the multiplicity of \( r_z \) must not greater than two.

In order to find the zero-stability of the block (13), we only consider the solution of the first characteristic polynomial, That is

\[
\Pi(r) = | r I - \bar{B}^{[1]} | = | r \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix} - \begin{bmatrix}
0 & 1 & 0 & hs \\
0 & 1 & 0 & h \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
\end{bmatrix} | = r^2 (r - 1)^2
\]

\[
= r^2 (r - 1)^2
\]
which implies $r = 0, 0, 1, 1$. Hence, our method is zero stable for all $s \in (0, 1)$

### 3.3 Consistency

The one step hybrid block method (13) is said to be consistent if its order greater than or equal one i.e. $P \geq 1$

This proves that our method is consistent for all $s \in (a, b)$.

### 3.4 Convergence

**Theorem 3.1** (Henrici, 1962). **Consistency and zero stability are sufficient conditions for a linear multistep method to be convergent**

Since the method is consistent and zero stable, it implies the method is convergent for all $s$.

### 3.5 Region of Absolute Stability

In this work, the region of absolute stability is defined by locus boundary method. The method (13) is said to be absolutely stable if for a given $h$, all roots of the characteristic polynomial $\pi(z, h) = \rho(z) - \bar{h}\sigma(z)$, satisfies $|z_i| < 1$. Test equation $y'' = -\lambda^2 y$ is substituted in (13) where $\bar{h} = -\lambda^2 h^2$ and $\lambda = \frac{df}{dy}$.

Then, substituting $z = e^{i\theta} = \cos\theta - i\sin\theta$ in the characteristic function and considering real part yields

$$\bar{h}(\theta, h) = \frac{(72\cos(\theta) - 72)}{(s^2\cos(\theta) - 3s + 2s^2)} \quad (20)$$

### 3.6 Numerical Results

By substituting $s = \frac{1}{3}$ into equation (14)-(17), the following block of one step with one hybrid points and its derivative are obtained

$$y_{n+\frac{1}{3}} = y_n + \frac{h}{3}y'_n + \left[ \frac{11}{324}f_n - \frac{1}{648}f_{n+1} + \frac{5}{216}f_{n+\frac{1}{3}} \right]$$

$$y_{n+1} = y_n + hy'_n + h^2 \left[ \frac{1}{12}f_n + \frac{1}{24}f_{n+1} + \frac{3}{8}f_{n+\frac{1}{3}} \right]$$

$$y'_{n+\frac{1}{3}} = y'_n + h \left[ \frac{4}{27}f_n - \frac{1}{108}f_{n+1} + \frac{7}{36}f_{n+\frac{1}{3}} \right]$$

$$y'_{n+1} = y'_n + h \left[ \frac{1}{4}f_{n+1} + \frac{3}{4}f_{n+\frac{1}{3}} \right] \quad (21)$$
Based on the approach used in section (3.1), the block and its derivative above are of order $[3,3,3,3]^T$ with error constant
In order to find the region of absolute stability of (21), $s = \frac{1}{3}$ is substituted into equation (20), and this gives
\[
\tilde{h}(\theta, h) = \frac{(648(\cos(t) - 1))}{(\cos(t) - 7)}
\]
Equation (22)
Evaluating equation (22) at intervals of $30^\circ$ produces results tabulated below.

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>0</th>
<th>30$^\circ$</th>
<th>60$^\circ$</th>
<th>90$^\circ$</th>
<th>120$^\circ$</th>
<th>150$^\circ$</th>
<th>180$^\circ$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{h}(\theta, h)$</td>
<td>0</td>
<td>14.1532</td>
<td>49.8462</td>
<td>92.5714</td>
<td>129.6</td>
<td>153.7224</td>
<td>162</td>
</tr>
</tbody>
</table>

Hence, the interval of absolute stability is $(0, 162)$ as demonstrated in the Figure 1.

Figure 1: Region stability of new method

In finding the accuracy of our methods, the following second order ODEs are examined. The new block methods solved the same problems the existing methods solved in order to compare results in terms of error.

**Problem 1:** \( y'' - x(y')^2 = 0, \ y(0) = 1, \ y'(0) = \frac{1}{2}, \ h = \frac{1}{320}. \)
Exact solution: \( y(x) = 1 + \frac{1}{2} \ln \left( \frac{2 + x}{2 - x} \right) \)

**Problem 2:** \( y'' - y' = 0, \ y(0) = 0, \ y'(0) = -1, \ h = 0.1. \)
Exact solution: \( y(x) = 1 - e^x \)
Table 1: Comparison of the new method with Awoyemi et al. (2011) for solving problem one, \( s = \frac{1}{3}, h = \frac{1}{320} \).

<table>
<thead>
<tr>
<th>x</th>
<th>exact solution</th>
<th>computed solution in our method with one off-step points ( s = \frac{1}{3} )</th>
<th>error in our method, ( P = 3 )</th>
<th>errors in Awoyemi et al. (2011), ( P = 4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>1.0500417292784914</td>
<td>1.0500417292790070</td>
<td>5.155876e-13</td>
<td>6.5650e-14</td>
</tr>
<tr>
<td>0.2</td>
<td>1.1003353477310753</td>
<td>1.1003353477301015</td>
<td>2.026157e-12</td>
<td>5.4803e-10</td>
</tr>
<tr>
<td>0.3</td>
<td>1.1511404359364665</td>
<td>1.1511404359408095</td>
<td>4.342970e-12</td>
<td>1.9256e-9</td>
</tr>
<tr>
<td>0.4</td>
<td>1.202735540540816</td>
<td>1.202735540611709</td>
<td>7.089218e-12</td>
<td>4.8029e-9</td>
</tr>
<tr>
<td>0.5</td>
<td>1.2554128118829946</td>
<td>1.2554128118925869</td>
<td>9.592327e-12</td>
<td>1.0006e-8</td>
</tr>
<tr>
<td>0.6</td>
<td>1.3095196042031119</td>
<td>1.3095196042137909</td>
<td>1.067901e-11</td>
<td>1.8727e-8</td>
</tr>
<tr>
<td>0.7</td>
<td>1.3654437542713971</td>
<td>1.3654437542797029</td>
<td>8.305800e-12</td>
<td>3.2346e-8</td>
</tr>
<tr>
<td>0.8</td>
<td>1.423649301936035</td>
<td>1.423649301924546</td>
<td>1.148859e-12</td>
<td>5.3969e-8</td>
</tr>
<tr>
<td>0.9</td>
<td>1.487002785940546</td>
<td>1.487002785698400</td>
<td>2.421463e-11</td>
<td>8.8004e-8</td>
</tr>
<tr>
<td>1.0</td>
<td>1.5493061443340586</td>
<td>1.5493061442610176</td>
<td>7.304091e-11</td>
<td>1.4353e-7</td>
</tr>
</tbody>
</table>

Table 2: Comparison of the new method with Kayode and Adeyeye (2013) for solving Problem 2, \( s = \frac{1}{3}, h = \frac{1}{10} \).

<table>
<thead>
<tr>
<th>x</th>
<th>exact solution</th>
<th>computed solution in our method with one off-step points ( s = \frac{1}{3} )</th>
<th>error in our method, ( P = 3 )</th>
<th>errors in Kayode and Adeyeye (2013), ( P = 4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>-0.10517091807564771</td>
<td>-0.10517092711226853</td>
<td>9.036621e-9</td>
<td>8.17176e-7</td>
</tr>
<tr>
<td>0.2</td>
<td>-0.22140275816016985</td>
<td>-0.22140277922667850</td>
<td>2.106651e-8</td>
<td>3.10356e-6</td>
</tr>
<tr>
<td>0.3</td>
<td>-0.349855880757600318</td>
<td>-0.34985588441953232</td>
<td>3.661932e-8</td>
<td>6.50957e-6</td>
</tr>
<tr>
<td>0.4</td>
<td>-0.4918246976417035</td>
<td>-0.4918247539427932</td>
<td>5.630301e-8</td>
<td>1.14380e-5</td>
</tr>
<tr>
<td>0.5</td>
<td>-0.64872127070012819</td>
<td>-0.64872135151457455</td>
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4 Conclusion

A one step hybrid block method with one generalized off step points was formulated. The method was tested to be convergent with order three for all off step point belong to selected interval. The developed method was applied to solve both non-linear and linear problems of second ODEs without reduction to the equivalents system of first order ODEs. Generated results confirm the high accuracy of the new methods.

References


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