Sensitivity of Unstable Eigenmodes to Parameters in a Linearized Geophysical Flow Model

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Abstract

The influence of variations in the parameters of a stratified rotating geophysical flow model on the growth rates of baroclinic unstable eigenmodes is explored using a sensitivity analysis. Analytical expressions for sensitivity coefficients are obtained that allow for determining both the conditions for instability and main characteristics of unstable normal modes.

Mathematics Subject Classification: 76B60, 86A10

Keywords: normal modes, eigenmodes, sensitivity analysis, baroclinic instability

1 Introduction

The term “geophysical flows” refers to large scale naturally occurring motions in the atmosphere, ocean, lithosphere and outer core of the Earth and other planets. Geophysical flows contain a complex mixture of propagating eddies and stable and unstable waves, but stratification and rotation of the fluid play essential role in the dynamics of geophysical flows [12, 18]. Stability studies of these flows and determination of properties of stable and unstable waves are very important in many areas including the large-scale circulation of the atmosphere and ocean. One particular, but very important, form of instability that can occur in a rotating and stratified fluid is baroclinic instability (BI) [2, 3]. The mechanism of BI has signi-
ficant geophysical applications [7, 9, 12, 14, 18]. In particular, BI determines the genesis of large-scale eddies in the atmosphere, which significantly affect the weather conditions over large geographical regions and play a considerable role in the atmospheric general circulation, effecting the meridional and vertical transport of heat, water vapor and momentum and in such a way smoothing and leveling the temperature contrasts between high and low latitudes [11].

In the atmosphere, BI is commonly considered as the instability of a westerly unperturbed flow and can be examined as an eigenvalue problem using linearized dynamics equations [7, 12, 14]. A solution of this problem allows us to determine the conditions for instability and characteristics of unstable eigenmodes such as their initial growth rates and phase speeds. The development of BI in the atmosphere depends primarily on the meridional temperature gradient (MTG), in particular at the surface and static stability (temperature lapse rate) [7]. Observed climate changes affect the atmospheric thermal structure and dynamics and influence the atmospheric static stability and the MTG [1, 5, 6, 8, 10, 14]. As a result, favorable conditions for BI can be also altered [17]. To evaluate how climate change will affect the large-scale wave dynamics in the atmosphere and its general circulation, it is necessary to consider the influence of variations in MTG and static stability on the development of BI. In our previous paper [17] the impact of climate change on the initial growth of unstable baroclinic eigenmodes was theoretically explored via the Eady-type atmospheric model [3] that is one of the most simplified mathematical models of baroclinic instability. In this paper, we will employ a more realistic multilayer atmospheric model to study the response of the initial growth rates of baroclinic unstable modes to variations in the static stability parameter and the wind vertical shear, which are taken as the indicators of climate change. Results obtained may be generalized to dynamic stability in wide class of large-scale sheared, rotating and stratified geophysical flows.

2 The Model

The model equations consist of quasi-geostrophic vorticity equation and hydrostatic thermodynamic equation, which have been derived many times in the scientific literature (e.g. [7]). Applying pressure as the vertical coordinate and the mid-latitude \( \beta \)-plane approximation, these equations can be written as follows:

\[
\begin{align*}
\nabla^2 \partial_t \psi + J(\psi, \nabla^2 \psi) + \beta \partial_x \psi - f_0 \partial_p \omega &= 0 \\
\partial_p^2 \psi + J(\psi, \partial_p \psi) + \sigma \omega / f_0 &= 0
\end{align*}
\]

where \( \psi \) is the stream function, \( \sigma(p) \) is the static stability parameter, \( f_0 \) is a standard value of the Coriolis parameter, \( \omega \) is the pressure vertical velocity, and \( J(a, b) = \partial_x a \partial_y b - \partial_y a \partial_x b \) is the Jacobian operator. The vertical boundary conditions come from the assumption that the vertical velocity vanishes at the top.
Sensitivity of unstable eigenmodes

\( p = 0 \) and the bottom \( (p = 10^5 \text{ Pa}) \) of the atmosphere:

\[
\omega = 0 \text{ at } p = 0 \text{ and } p = 10^5 \text{ Pa}
\] (3)

We assume the periodic boundary conditions in both the \( x \) and \( y \) directions. The basic flow is assumed to be zonal and eastward and defined by the temperature profile \( T = T(y, p) \) via the thermal wind relationship. Taking into account the hydrostatic equation and relations between \( \Psi \) and the components of wind speed \( u \) and \( v \), the zonal flow can be specified as:

\[
\bar{u}(y, p) = -\partial_p \psi, \quad \partial_p \bar{u} = -\partial_{yp} \psi, \quad \bar{v} = -\partial_x \psi = 0, \quad \bar{\omega} = 0
\] (4)

A bar over symbols is used to indicate the zonally averaged values of variables characterizing the basic flow. Applying the technique of separation of variables, let us represent the model state variables \( \Phi \) and \( \omega \) in the form of eigenmodes (normal modes):

\[
\psi(x, y, p, t) = -\bar{u}(p)y + \hat{\psi}(p, t)e^{i(kx+ny)}
\]

\[
\omega(x, y, p, t) = \hat{\omega}(p, t)e^{i(kx+ny)}
\] (5)

where \( \hat{\psi} \) and \( \hat{\omega} \) are complex perturbation amplitudes of the stream function and vertical velocity, and \( k \) and \( n \) are zonal and meridional wavenumbers. Substituting (5) into (1) and (2) yields the following linear ordinary differential equations:

\[
\partial_t \hat{\psi} = -ik\bar{u}\hat{\psi} + (k^2 + n^2)^{-1}(ik\beta\hat{\psi} - f_0 \partial_p \hat{\omega})
\] (6)

\[
\partial_{pt} \hat{\psi} = ik(\bar{u}'\hat{\psi} - \bar{u}\partial_p \hat{\psi}) - \sigma\hat{\omega}/f_0
\] (7)

where \( \bar{u}' = d\bar{u}/dp \). By differentiating the equation (6) with respect to \( p \), one can eliminate the time derivative from (7) to obtain the diagnostic \( \omega \)-equation:

\[
\partial_{pp} \hat{\omega} - (k^2 + n^2)\sigma\hat{\omega}/f_0^2 = (ik/f_0)[\beta\partial_p \hat{\psi} - 2(k^2 + n^2)\bar{u}'\hat{\psi}]
\] (8)

Equations (6) and (8) form the complete system of equations that allow us numerically exploring the baroclinic instability. Meanwhile, to study the influence of variations in the model parameters on the development of instability we can use sensitivity analysis together with eigenanalysis. Let us ignore the \( y \)-dependence of the zonal wind and the initial perturbations, i.e. we will consider only the baroclinic mechanism of the instability. By differentiating the equation (6) with respect to \( p \) and then eliminating \( \partial_{pp} \hat{\omega} \) as well as assuming that \( \hat{\psi}(p, t) = \hat{\psi}(p)e^{-ikt} \), where \( c \) is the complex phase velocity, we obtain the linearized potential vorticity equation:
\[ (c - \bar{u})[k^2 - f_0^2 \partial_p (\sigma^{-1} \partial_p)]\hat{\psi} + (\beta - f_0^2 \sigma^{-1} \bar{u}')\hat{\psi} = 0 \]  \hspace{1cm} (9)

with the boundary conditions

\[ (c - \bar{u})\partial_p \hat{\psi} + \bar{u}'\hat{\psi} = 0 \text{ at } p = 0 \text{ and } p = 10^5 \text{ Pa} \]  \hspace{1cm} (10)

Suppose that the model has \( N \) vertical layers. Approximating the derivatives in (9) and (10) by finite differences, we can obtain the set of homogeneous equations for \( \hat{\psi} \):

\[ (A - cB)\hat{\psi} = 0 \]  \hspace{1cm} (11)

where \( A \) and \( B \) are square matrices of order \( N \). The \( N \) eigenvalues are determined by the following condition:

\[ \det(B^{-1}A - cI)\hat{\psi} = 0 \]

where \( I \) is the identity matrix. The basic flow is unstable if \( c_i = \text{Im}(c) > 0 \), then the growth rate of unstable mode \( \chi_k \) corresponding to wavenumber \( k \) is calculated as \( \chi_k = kc_i \). For a small vertical resolution (\( N = 2 \)) the eigenvalue problem can be solved analytically giving us also analytical expressions for the sensitivity coefficients (see below). In two layer model the vorticity equation (1) is applied to both \( 0.75 \) and \( 0.25 \times 10^5 \) Pa surfaces (levels 1 and 2 respectively) and the hydrostatic thermodynamic energy equation (2) is written at \( 0.5 \times 10^5 \) Pa surface (level 1½). For a normal mode solution we can finally obtain the following expression for the phase speed of baroclinic eigenmodes:

\[ c = u_m - \frac{\beta}{k^2} \frac{k^2 + \lambda^2}{k^2 + 2\lambda^2} \pm \frac{\beta^2 \lambda^4}{k^4(k^2 + 2\lambda^2)} + \frac{u_T^2 k^2 - 2\lambda^2}{k^2 + 2\lambda^2} \]  \hspace{1cm} (12)

where \( (u_1 + u_2)/2 \) is the mean speed of basic flow, \( u_T = (u_1 - u_2)/2 \) is the wind shear, and \( \lambda^2 = f_0^2 [\sigma(\Delta p)]^{-1} \). Infinitesimal perturbations will grow exponentially if \( \delta < 0 \). Then the growth rate is given by the following expression:

\[ \chi_k \equiv kc_i = \sqrt{\frac{\beta^2 \lambda^4 + u_T^2 k^4 (k^4 - 4\lambda^4)}{k(k^2 + 2\lambda^2)}} \]

Thus, the increment of growing perturbations depends on two control variables, namely the wind shear \( u_T \) associated with the pole-equator temperature gradient (MTG), and the variable \( \lambda \), which is a function of static stability \( \sigma \). A detailed analysis of two-layer model is discussed in a number of scientific publications (e.g. [7, 9, 13, 14]). Here we only remark that a mode is unstable if it length \( L_X \) satisfies the following double inequalities \( L_{SC} < L_X < L_{LC} \), where \( L_{SC} \) and \( L_{LC} \) are, respectively, short-wave and long-wave cut-offs.
3 Sensitivity of Unstable Eigenmodes to Parameters

To estimate the influence of variations in the control variables on the development of baroclinic instability we will use sensitivity coefficients [16], which are introduced as partial derivatives of the growth rate \( \chi_k \) with respect to parameters \( \sigma \) and \( u_T \):

\[
S_\sigma \equiv \frac{\partial \chi_k}{\partial \sigma}, \quad S_{u_T} \equiv \frac{\partial \chi_k}{\partial u_T}
\]

If \( \delta \sigma \) and \( \delta u_T \) are, respectively, small variations in \( \sigma \) and \( u_T \) caused by external forcing, the unstable-mode growth-rate variations \( \delta \chi_k \) induced by \( \delta \sigma \) and \( \delta u_T \) are estimated at the first approximation in the following way:

\[
\delta \chi_k(\delta \sigma) \equiv \chi_k(\sigma + \delta \sigma) - \chi_k(\sigma) \approx \delta \sigma \times S_\sigma \\
\delta \chi_k(\delta u_T) \equiv \chi_k(u_T + \delta u_T) - \chi_k(u_T) \approx \delta u_T \times S_{u_T}
\]

where sensitivity coefficients \( S_\sigma \) and \( S_{u_T} \) are calculated around some reference values of static stability parameter \( \sigma^* \) and thermal wind \( u_T^* \). Each pair \((\sigma^*, u_T^*)\) matches a certain reference climate. Growth rates of unstable eigenmodes can be calculated by means of the equation (12) for each particular reference climate, which makes it possible to estimate the influence of changes in reference values of \( \sigma^* \) and \( u_T^* \) on the development of baroclinic instability. However, we are interested in estimating the impact of infinitesimal perturbations of \( \sigma \) and \( u_T \) on the growth rate of unstable eigenmodes using the relations (14) and analytical expressions for sensitivity coefficients \( S_\sigma \) and \( S_{u_T} \). Differentiating the equation (12) with respect to \( \sigma \) and \( u_T \), we get

\[
S_\sigma = - \frac{k^2 \lambda^2}{\sigma (k^2 + 2\lambda^2)^2} \frac{2u_T^2 k^4 (k^4 + 2\lambda^2) - \beta^2 \lambda^2}{u_T^2 k^4 (k^4 - 4\lambda^4) + \beta^2 \lambda^4} \sqrt{\frac{u_T^2 k^4 (k^4 + 2\lambda^2)}{\beta^2 \lambda^4 + u_T^2 k^4 (k^4 - 4\lambda^4)}}
\]

\[
S_{u_T} = -u_T k^3 (k^2 - 2\lambda^2) \frac{\sqrt{\beta^2 \lambda^4 + u_T^2 k^4 (k^4 - 4\lambda^4)}}{\beta^2 \lambda^4 + u_T^2 k^4 (k^4 - 4\lambda^4)}
\]

Sensitivity coefficients \( S_\sigma \) calculated for several reference values of \( \sigma^* \) as functions of wavelength \( L_x \) are shown in Fig. 1, from which it can be seen that for a given \( \sigma^* \), the absolute value of \( S_\sigma \) exponentially increases in magnitude as the wavelength decreases. Hence, growth rates of shorter baroclinic waves are more sensitive with respect to static stability than growth rates of longer waves. Small departures of static stability from the reference values lead to changes in the growth rates \( \chi_k \), and the smaller the wavelength is, the larger changes are.
Figure 1: Absolute values of sensitivity coefficients $S_\sigma$ for different values of $\sigma^* \times 10^6$ m$^2$ Pa$^{-2}$ s$^{-2}$ for the case of $u^*_T = 7.5$ m s$^{-1}$.

Figure 2: Sensitivity coefficients $S_{u_T}$ for different values of thermal wind $u^*_T$ for the case of $\sigma^* = 2 \times 10^{-6}$ m$^2$ Pa$^{-2}$ s$^{-2}$.

As an example let us analyze in more details two eigenmodes of wavelengths $L_x^{(1)} \approx 3200$ and $L_x^{(2)} \approx 5000$ km for the case of $\sigma^* = 2 \times 10^{-6}$ m$^2$ Pa$^{-2}$ s$^{-2}$, $u^*_T = 7.5$ and $u_m = 15$ m s$^{-1}$. The sensitivity $S^{(1)}_\sigma = -10.60$ for a normal mode with $L_x^{(1)}$ is about 8 times the sensitivity $S^{(2)}_\sigma = -1.34$ for a mode with $L_x^{(2)}$ in absolute value. The growth rates of these modes are, respectively, $\chi^{(1)}(\sigma^*) \approx 0.21$ and $\chi^{(2)}(\sigma^*) \approx 0.37$ day$^{-1}$. A minus 5 percent change in static stability parameter increased the growth rates of these modes on $\Delta \chi^{(1)} \approx 0.13$ and $\Delta \chi^{(2)} \approx 0.01$ respectively. Thus, $\chi^{(1)}(\sigma^* - 0.05\sigma^*) \approx 0.34$ day$^{-1}$, but $\chi^{(2)}$ has changed slightly $\chi^{(2)}(\sigma^* - 0.05\sigma^*) \approx 0.38$ day$^{-1}$.
Sensitivity coefficients $S_{u_T}$ versus wavelength $L_x$ calculated for several reference values $u_T^*$ for the case of $\sigma^* = 2 \times 10^{-6} \text{ m}^2 \text{ Pa}^{-2} \text{ s}^{-2}$ are plotted in Fig. 2. This picture provides evidence of strongly nonlinear relationship between $S_{u_T}$ and $u_T^*$. Nonlinearity is particularly pronounced at the edges of the spectrum of unstable modes, however long modes are more sensitive to changes in the thermal wind than short modes. Because control parameters, $\sigma$ and $u_T$, have different units and very different orders of magnitude, to compare the effects of $\sigma$ and $u_T$ on the growth rates of unstable modes one should use normalized sensitivity coefficients $S_{\sigma}^R$ and $S_{u_T}^R$, referred to as relative sensitivity coefficients. These coefficients are introduced as follows:

$$S_{\sigma}^R = S_{\sigma}(\sigma/\chi_k), \quad S_{u_T}^R = S_{u_T}(u_T/\chi_k)$$

The graphs of $|S_{\sigma}^R|$ and $S_{u_T}^R$ are plotted in Fig. 3. The analysis of relative sensitivity coefficients leads to a very important conclusion. There is a critical value of wavelength $L_x^{cr}$ that divides the spectrum of unstable modes in two parts. For those unstable modes for which $L_x > L_x^{cr}$, the wind shear (i.e. MTG) plays a dominant role on the development of baroclinic instability. However, if $L_x < L_x^{cr}$ then the prevailing role belongs to the static stability. The critical value of wavelength is the value at which $|S_{\sigma}^R| = S_{u_T}^R$. For $\sigma^* = 2 \times 10^{-6} \text{ m}^2 \text{ Pa}^{-2} \text{ s}^{-2}$ and $u_T^* = 7.5 \text{ m} \text{s}^{-1}$ we obtain $L_x^{cr} \approx 3800$ km.

4 Concluding remarks

In this paper, we have considered the application of sensitivity analysis to estimate the response of growth rates of unstable eigenmodes of a stratified rotating
atmospheric flow to parameters that control the development of baroclinic instability, namely the static stability parameter and thermal wind, which indirectly reflects the horizontal baroclinicity (meridional temperature gradient). Analytical expressions for absolute and relative sensitivity coefficients were derived utilizing the two-layer quasi-geostrophic baroclinic model of the atmosphere on a beta-plane. Variations in the control parameters may be due to the observed climate change.

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Received: February 20, 2016; Published: April 1, 2016