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Traveling Wave Solutions for a Generalization of the (3+1)-Dimensional Kadomtsev-Petviashvili Equation with Variable Coefficients

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Abstract

We study a generalization of the (3+1)-dimensional Kadomtsev-Petviashvili (KP) equation from the point of view of its exact solutions. The equation have as particular case the standard (3+1)-dimensional KP equation. The relevance of the calculus made here is that, we can found exact solutions for others important models of nonlinear partial differential equation with higher-order nonlinearity such as the KdV equation, the modified KdV equation and other models used in dynamic of fluids where the coefficients depend on the time variable. We explain this fact by mean of one example.

Mathematics Subject Classification: 35C05

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1 Introduction

The analytic study of solutions for nonlinear partial differential equations with higher-order nonlinearity is a no easy task to made. Due the importance of many of this models, the analysis of these equations by mean of computational methods have relevance today. Many computational methods have been implemented for to solve this models, where one of the most important method is the extended tanh method [1]. In this paper, we will use the improved tanh-coth method [2] whose solutions give us expressions in more general form that the previous.

The main objective of the paper consists on obtain exact solutions to the following generalization of the (3+1)-dimensional Kadomtsev-Petviashvili equation with variable coefficients

$$(u_t + \alpha(t)u^n u_x + \beta(t)u_{xxx})_x + \rho(t)u_{yy} + \delta(t)u_{yy} = 0, \tag{1}$$

being $\alpha(t)$, $\beta(t)$, $\rho(t)$, $\delta(t)$ arbitrary functions depending of the time t, $u(x, y, z, t)$ the unknown function and n is a positive integer number. The (3+1)-dimensional Kadomtsev-Petviashvili equation which is obtained in the case $\alpha(t) = 6$, $\beta(t) = 1$, $\rho(t) = 3$ and $\delta(t) = 3$ have relevance particulary in some fields of the physics. For instance, this equation explains the dynamic of soliton and nonlinear waves in plasmas physics as well as in dynamic of fluids. For more details on applications, can be see [3] and references therein. The Painlevé Analysis and Bäcklund transformation have been made by the authors in [4] when (1) have the previous coefficients and, in this last case, exact solutions for (1) have been obtained in [5] using the extended tanh method. We will obtain exact solutions to Eq.(1) and we will observe that the solutions obtained can be used (using a brief change) as exact solutions of the following general equation [6]

$$u_t(x,t) + \alpha(t)u^n(x,t)u_x(x,t) + \beta(t)u_{xxx}(x,t) = 0. \tag{2}$$

The relevance of Eq.(2) consist on that two particular cases are the Korteweg-de Vries equation [7]

$$u_t(x,t) + 6u(x,t)u_x(x,t) + u_{xxx}(x,t) = 0, \tag{3}$$

and the modified Korteweg-de Vries equation [8]

$$u_t(x,t) + k_1 u^2(x,t)u_x(x,t) + u_{xxx}(x,t) = 0. \tag{4}$$

On the other hand, can be see in [9], [10] and references therein, that the use of coefficients depending of the variable t in (2) give us models used in dynamic of fluids. Two of this models are the following

$$u_t(x,t) + k_1 t^p u^2(x,t) u_x(x,t) + k_2 t^m u_{xxx}(x,t) = 0, \tag{5}$$

and

$$u_t(x,t) + k_1 u^n(x,t) u_x(x,t) + u_{xxx}(x,t) = 0. \tag{6}$$

2 Exact traveling wave solutions to Eq.(1)

We begin with the transformation

$$\begin{cases} u(x,y,z,t) = v(\xi) \\ \xi = x + y + z + \lambda t + \xi_0, \end{cases} \tag{7}$$

where ξ_0 is an arbitrary constant. Applying the transformation (7) to (1) and making one first integration with respect to ξ, we have the ordinary differential equation

$$(\lambda + \rho(t) + \delta(t))v'(\xi) + \alpha(t)v^n(\xi)v'(\xi) + \beta(t)v'''(\xi) = 0, \tag{8}$$

as we are searching exact solutions, we have taken the integration constant as zero. With one more integration we obtain

$$(\lambda + \rho(t) + \delta(t))v(\xi) + \frac{\alpha(t)}{n+1} v(\xi)^{n+1} + \beta(t)v''(\xi) = 0. \tag{9}$$

where, as in the previous case, we have taken the integration constant as zero. By using the transformation

$$v(\xi) = V^{\frac{1}{n}}(\xi), \tag{10}$$

then Eq.(9) reduces to

$$\begin{cases} (\lambda + \rho(t) + \delta(t))n^2(n+1)V^2(\xi) + \alpha(t)n^2V^3(\xi) \\ +\beta(t)[(1-n^2)V'(xi)^2 + n(n+1)V(\xi)V''(\xi)] = 0. \end{cases} \tag{11}$$

According with the improved tanh-coth method, we seek a solution to (11) using the expansion

$$V(\xi) = \sum_{i=0}^{M} a_i(t)\phi(\xi)^i + \sum_{i=M+1}^{2M} a_i(t)\phi(\xi)^{M-i}, \tag{12}$$

where M is a positive integer determined by balancing method. The new variable $\phi(\xi)$ is solution of the generalized Riccati equation

$$\phi'(\xi) = r(t) + q(t)\phi(\xi) + p(t)\phi(\xi)^2, \tag{13}$$

the $a_i(t)$, $i = 1, 2, \ldots, 2M$, $r(t)$, $p(t)$, $q(t)$ are functions depending only of the variable t to be determined later. The solution of (13) in the case $q(t)^2 - 4p(t)q(t) \neq 0$ is given by (see [11]):

$$\phi(\xi) = \frac{-\sqrt{q(t)^2 - 4p(t)r(t)}\tanh[\frac{1}{2}\sqrt{q(t)^2 - 4p(t)q(t)}\xi + \xi_0] - q(t)}{2p(t)}. \quad (14)$$

Substituting (12) into (11), and balancing $V(\xi)V''(\xi)$ with $V(\xi)^3$ we obtain

$$2M + 2 = 3M,$$

so that

$$M = 2.$$

Therefore, (12) reduces to

$$V(\xi) = a_0(t) + a_1(t)\phi(\xi) + a_2(t)(\phi(\xi))^2 + a_3(t)(\phi(\xi))^{-1} + a_4(t)(\phi(\xi))^{-2}. \quad (15)$$

The substitution of (15) along with (13) into (11) leads to an algebraic system in the unknowns $a_0(t)$, $a_1(t)$, $a_2(t)$, $a_3(t)$, $a_4(t)$, $r(t)$, $q(t)$, $p(t)$, which by reasons of space, we omit here. Solving the system with aid of the Mathematica we obtain a lot of solutions, from which, for the sake of simplicity we put here the most general:

$$\begin{cases} r(t) = \frac{n^2(\rho(t)+\delta(t)+\lambda)}{16\beta(t)p(t)}, \quad a_0(t) = -\frac{(n^2+3n+2)(\rho(t)+\delta(t)+\lambda)}{4\alpha(t)} \\ a_2(t) = -\frac{2\beta(t)p(t)^2(n^2+3n+2)}{\alpha(t)n^2}, \quad a_4(t) = -\frac{n^2(n^2+3n+2)(\rho(t)+\delta(t)+\lambda)^2}{128\alpha(t)\beta(t)p(t)^2} \\ a_1(t) = a_3(t) = q(t) = 0. \end{cases} \quad (16)$$

With respect to this values, and taking into account (14) we have

$$\phi(\xi) = -\frac{\sqrt{-\frac{n^2(\rho(t)+\delta(t)+\lambda)}{\beta(t)}}\tanh\left(\frac{1}{4}\sqrt{-\frac{n^2(\rho(t)+\delta(t)+\lambda)}{\beta(t)}}(\xi)\right)}{4p(t)}. \quad (17)$$

Finally, by (15), (10) and (7) we have that one solution of (1) is given by

$$u(x, y, z, t) = v(\xi) = \left[a_0(t) + a_2(t)\phi(\xi)^2 + a_4(t)\phi(\xi)^{-2}\right]^{\frac{1}{n}}, \quad (18)$$

where $a_0(t)$, $a_2(t)$, $a_4(t)$ are given by (16), $\phi(\xi)$ given by (17), $p(t)$, arbitrary function in the variable t, ξ given by (7), where ξ_0 and λ are arbitrary parameters.

We observe that if we take $\rho(t) = \delta(t) = 0$ in Eq.(1) and we consider $u = u(x,t)$, after integration with respect to variable x and taking the integration constant as zero we obtain Eq.(2). We can use the transformation

$$\begin{cases} u(x,t) = v(\xi) \\ \xi = x + \lambda t + \xi_0, \end{cases} \tag{19}$$

in which case Eq.(2) reduces to Eq.(8). Therefore, the solutions obtained for (1) can be used as exact traveling solutions to (2) using the variable ξ given by (19). Clearly, the equations (3), (4), (5), (6) are particular cases of (2). So that, if we take $\rho(t) = \delta(t) = 0$, $\alpha(t) = k_1 t^p$, $\beta(t) = k_2 t^m$ and $n = 2$ in (16) and (17), then

$$u(x,y) = v(\xi) = \left[a_0(t) + a_2(t)\phi(\xi)^2 + a_4(t)\phi(\xi)^{-2}\right]^{\frac{1}{2}}, \tag{20}$$

where ξ is given by (19), is solution to (5).

3 Conclusions

We have obtained exact solutions to a generalization of the (3+1)-dimensional Kadomtsev-Petviashvili (KP) equation by using the improved tanh-coth method. The relevance of the calculus made here consists on the fact that the obtained solutions are in a more general form that those derived by other authors in the case of the classical KP equation. However, the solutions derived in this paper, with a brief changes, can be used as solutions for others non-lineal model as the KdV and Modified KdV equations as well as others important models with variable coefficients used in dynamic of fluids. Clearly, the variation of the parameter ξ_0 give us a variety of soliton and periodic type solutions.

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