Mathematical Model of the Plate on Elastic Foundation Interacting with Pulsating Viscous Liquid Layer

L.I. Mogilevich and V.S. Popov
Department of Applied Mathematics
Yuri Gagarin State Technical University of Saratov, Russian Federation

L.N. Rabinsky and E.L. Kuznetsova
Department of a Resistance of Materials, Dynamics and Strength of Machines
Moscow Aviation Institute (National Research University)
Moscow, Russian Federation

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Abstract

The problem of dynamic interaction of the plate on the elastic foundation with pulsating viscous incompressible liquid layer is set up and analytically solved. The problem in a flat setting for the regime of a stationary pulsating liquid movement in the canal under the suggested difference of pressure at its butt end is considered. Deflections of plate are modeled as one-mass system elastic displacements. Parameters of one-mass system were found by using method of equivalent mass. The formulated boundary problem represents non-linear Navier-Stocks equations system for viscous incompressible liquid layer and the equation of one-mass system. The conditions of liquid adhesion to impenetrable channel walls and the condition of free leakage of liquid at channel butt ends are presented in the paper as the boundary ones. The linearization of the problem by means of perturbation method is made. The solution of the linearized problem is made by means of the assigned forms method for adjusted harmonic fluctuations.

Keywords: mathematical modeling, hydroelasticity, viscous liquid, elastic foundation, pressure pulsating, fluctuations, plate, one-mass system
Introduction

The investigations of hydroelastic problems present one of the contemporary trends of mechanics and applied mathematics development. In the context of this trend, the problems of flat channel oscillating elastic walls filled with liquid can be singled out. The dynamics of viscous incompressible liquid layer in the channel with absolutely solid walls is investigated in the frames of hydrodynamic lubrication theory [1, 2]. Paper [3] presents the experimental study of periodically pulsating viscous liquid movement stability in flat channel with immovable solid walls. The oscillations of a plate plunged in an ideal incompressible liquid with a three surface are investigated by M.R. Haddara and S. Cao [4]. The problem of bending oscillations of a channel wall as a beam-strip interacting with an ideal liquid, filling the channel is solved by D.A. Indeitsev et al. [5]. However, this approach does not allow damping in the considered oscillating system because of the liquid viscosity disregard. The hydroelastic oscillations of cantilevered beam in an unlimited volume of viscous incompressible liquid with an initial set of linearized boundary conditions on an unperturbed beam surface are investigated by Cassio T. Faria and Daniel J. Inman [6]. On the other hand, the plate chaotic oscillations in the course of its interaction with an ideal incompressible fluid considering the boundary conditions on perturbed plate surface are investigated by K.V. Avramov and E.A. Strel’nikova [7]. The damping harmonically vibrating infinitely long beam-strip on the viscous incompressible and compressible liquid are studied by M. Enelund [8] and T. Önsay [9, 10]. The interaction of vibrating discs with viscous incompressible liquid layer between them is investigated by L.I. Mogilevich and V.S. Popov [11]. In so doing, the elastic oscillations of the channel walls at resonance frequencies lead to the change of liquid pressure tens of times and cause vibration cavitation. The analogous problem for the two vibrating plates is solved by L.I. Mogilevich et al. [12]. M. Amabili [13] has studied the vibration of a round plate on a free surface of an ideal incompressible liquid. In so doing, the liquid section limited by a rigid bottom and cylindrical surface is considered. The oscillations of a round plate, plunged into water with free surface are investigated by E. Askari et al. [14]. T.D. Akcabay and Y.L. Young [15] carried out the study of hydroelastic beam oscillations in a viscous flow in application to piezoelectric element, which may be used for receiving energy from the flow. R.V. Ageev et al. [16] set up and analytically solved the problem of bending hydroelastic oscillations of a homogeneous plate, forming the wall of a slit channel with a pulsating layer of viscous incompressible liquid with an assigned harmonic law of pressure pulsating at one butt end. The interaction of the elastic cylinder with viscous incompressible liquid layer on the basis of the one-mass system and the method of equivalent mass is investigated by L.I. Mogilevich and V.S. Popov [17].

The investigation of the oscillation thin-walled elements on elastic foundation interacting with a viscous liquid layer is of theoretical importance, while its results are of practical interest for computing and analyzing the new technology systems and objects. But there is shortage of studies devoted to the interaction of
Mathematical model of the plate on elastic foundation

It may be stated, that V.V. Alekseev et al. [18] presented the first studies of this issue. They consider the oscillations of the channel wall (elastic membrane on elastic foundation) on the bottom of a tank containing a heavy ideal liquid. However, the problems of thin-walled elements on elastic foundation interacting with a viscous liquid are not investigated in the abovementioned sources.

Mathematical formulation

Let us consider the channel in the fig.1. It is formed by two parallel walls 1, 2 of identical geometric sizes with a pulsating layer of viscous incompressible liquid 3, moving at the expense of the suggested difference of pressure at the butt ends. The wall 1 is a plate on Winkler elastic foundation. The plate is simply supported at the butt ends. The wall 2 is an absolutely solid body. The geometrical size of the channel $2\ell$ is considerably smaller than it’s size $b$, and the liquid layer thickness (the distance between the walls) represents an undisturbed state $\delta_0$ and is considerably smaller than $2\ell$. As a result of the pressure pulsating, there emerge the wall 1 fluctuations, its amplitude shift being considerably smaller than $\delta_0$. The leakage at butt ends can be considered as a spurt one in the cavity, filled with the same liquid. To be definite, let us consider, that the pressure in a left cavity is $p^+(\omega t) + \Delta p$, and in a right cavity is $p^+(\omega t)$ (component of pressure, which harmonically changing in time, $\Delta p$ - constant difference of pressure at the butt ends).

![Fig. 1. The scheme of the flat channel with a wall-plate on elastic foundation](image)

The law of component of pressure $p^+(\omega t)$ at the butt end is presented as:

$$p^+(\omega t) = p_m^+ f_p(\omega t), \quad f_p(\omega t) = \exp(i\omega t),$$

where $p_m^+$ is the amplitude of pressure pulsating at butt ends; $\omega$ is the pulsating frequency; $f_p(\omega t)$ is the pressure change law.

Later, we take into account, that in a given mechanical system strong damping determined by liquid viscosity record, takes place. In this case, the influence of initial conditions ceases to influence and harmonic oscillations emerge [19, 20].
Therefore, from the very beginning of the research in lengthy processes of general solution of heterogeneous equations, initial conditions will be excluded.

Let us introduce the Cartesian coordinates system \( x, y, z \), connected with a wall \( J \). Taking into account, that \( 2F << b \), we consider the channel to be unlimited in axis direction \( y \), that is we start solving a flat problem. The deflections of plate are modeled as displacements of one-mass system. Its parameters are defined by equivalent mass method [17] by taking deflection form as trigonometric functions. In this case dynamic equation of one-mass system is as follows

\[
m_1 \frac{d^2 z_1}{dt^2} + n_1 z_1 = N, \tag{2}
\]

where \( n_1 = Db\pi^5\ell^{-3}(1 + (\kappa/D)(2\ell/\pi)^4)/16 \) is the equivalent rigidity of elastic suspension, \( m_1 = \rho_1 h_1 b \ell(1 + (1/4)(1/16 + (\kappa/D)(2\ell/\pi)^4)/(1 + (\kappa/D)(2\ell/\pi)^4)) \) is the equivalent plate mass, \( D \) is the plate bending stiffness, \( \kappa \) is the elastic coefficient of Winkler foundation, \( h_0 \) is the plate thickness, \( \rho_0 \) is the plate density, \( N \) is force, acting from liquid pulsating layer. The expression for the force \( N \) takes the form of

\[
N = \int_{0-t} q_c dx dy, \quad z = z_{1m} f(\omega t), \tag{3}
\]

where \( q_c = -p + 2\rho \nu (\partial V_c/\partial z) \) is the normal tension, acting from the liquid pulsating layer on the channel wall.

The dynamic equations of viscous incompressible liquid in the channel are

\[
\frac{\partial V_x}{\partial t} + V_x \frac{\partial V_x}{\partial x} + V_z \frac{\partial V_x}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left( \frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial z^2} \right), \tag{4}
\]

\[
\frac{\partial V_z}{\partial t} + V_x \frac{\partial V_z}{\partial x} + V_z \frac{\partial V_z}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \left( \frac{\partial^2 V_z}{\partial x^2} + \frac{\partial^2 V_z}{\partial z^2} \right), \quad \frac{\partial V_x}{\partial x} + \frac{\partial V_z}{\partial z} = 0,
\]

where \( p \) – pressure, \( \rho \), \( \nu \) – density and kinematical coefficient of the liquid viscosity, \( V_x, V_z \) – velocity movement projections on coordinates axis.

The liquid dynamic equations are expanded by the boundary conditions: liquid adhesion to channel walls [16, 17, 20, 21]

\[
V_x = 0, \quad V_z = 0 \quad \text{at} \quad z = z_0, \quad V_x = 0, \quad V_z = dz_1/dt \quad \text{at} \quad z = z_{1m} f(\omega t), \tag{5}
\]

and the boundary conditions of its free butt ends leakage

\[
p = p^+(\omega t) \quad \text{at} \quad x = \ell, \quad p = p^+(\omega t) + \Delta p \quad \text{at} \quad x = -\ell. \tag{6}
\]

Here \( z_1 = z_0 + z_{1m} f(\omega t) \) is the law of the channel wall shift; \( z_0 \) is a static wall shift at the expense of static pressure difference \( \Delta p \); \( z_{1m} \) is the oscillation amplitude of the channel wall at the expense of pulsating pressure \( p^+(\omega t) \).

**Solutions and discussions**

Let us introduce in our research the dimensionless variables [16, 20]

\[
\psi = \delta_0/\ell << 1, \quad \lambda = z_{1m}/\delta_0 << 1, \quad \tau = \omega t, \quad \xi = x/\ell, \quad \zeta = z/\delta_0, \quad V_z = z_{1m} \omega U_z, \tag{7}
\]

\[
V_x = z_{1m} \omega U_z/\psi, \quad p = p^+(\tau) + \langle z_{1m} \rho \nu \alpha (\delta_0^2 \psi^2)^{-1} P \rangle, \quad Re = \delta_0^2 \omega/\nu, \quad \Delta p = z_{1m} \rho \nu \alpha (\delta_0^2 \psi^2)^{-1} \Delta P.
\]
Here $\psi$, $\lambda$, $\text{Re}$ are the parameters, describing the problem.

In the suggested problem setting $\psi$ is a dimensionless small parameter and therefore in a zero approximation with $\psi$ the equations (2), (4) are simplified. Then while considering asymptotic expansions by a small parameter $\lambda \ll 1$ [20-22] $P = P_0 + \lambda P_1 + \ldots, U_\xi = U_{\xi 0} + \lambda U_{\xi 1} + \ldots$ and limiting by only one member of the expansion, we get the hydroelasticity linearized problem, including the equations of liquid layer dynamics

$$\text{Re} \partial U_{\xi 0}/\partial \tau = -\partial P_0/\partial \xi + \partial^2 U_{\xi 0}/\partial \xi^2, \quad \partial P_0/\partial \xi = 0, \quad \partial U_{\xi 0}/\partial \xi + \partial U_{\xi 0}/\partial \xi = 0$$

with boundary conditions

$$U_{\xi 0} = 0, \quad P_0 = \Delta P \quad \text{at} \quad \xi = 0, \quad U_{\xi 0} = 0, \quad P_0 = 0 \quad \text{at} \quad \xi = 1,$$

and the equation of wall

$$m_t d^2 z_t/df^2 + n_t z_t = -2b\ell p^*(r) - b\ell \rho v_{3w}(\delta_{5\nu}^2)^3/3 \int_1^1 P_0 d\xi.$$

The solution of the equations system (8) with boundary conditions (9), (10) in the suggestion of the harmonic character time change of hydrodynamic parameters and channel wall shift are found in the form of

$$U_{\xi 0} = \frac{1}{2\ell^2} \left[ \frac{\partial^3 P_0}{\partial \xi^3} (1 + \Phi(\xi)) + \frac{\partial P_0}{\partial \xi} \Phi(\xi) \right],$$

$$U_{\xi 0} = \frac{1}{2\ell^2} \left[ \frac{\partial^3 P_0}{\partial \xi^3} (\Phi(\xi) - \Phi(\xi)') \right],$$

$$P_0 = [(1 - \xi^2)(2\ell^2 \chi(\omega) d^2 f/d df^2 + 12\gamma(\omega) d^2 f/d df + \Delta P(1 - \xi^2))/2],$$

where $\gamma(\omega) = \sqrt{\text{Re}/2}$, $\Phi(\xi) = F_1(\alpha^2) D_1 - F_2(\alpha^2) D_2 + 2 D_3(\alpha^2) D_4$, $\Phi(\xi) = \int_0^\Phi(\xi) d\xi$, $\Phi(\xi) = \int_0^\Phi(\xi) d\xi$.

The equation (2) in consideration of (12), (7) take the form of

$$(m_t + M) d^2 z_t/df^2 + 2K d z_t/df + n_t z_t = -b \ell \Delta p - 2b p^*(\omega t),$$

where $2K = 8b\ell\rho v(\delta_5^2)^{-1}, M = 4b\rho v(\omega \delta_5^2)^{-1} \delta^2 \omega/3$.

The solution of the equation (13) under the law of pressure change in time (1) takes the form of

$$z_1 = z_0 + z_m f(\tau) = -b \ell \Delta p/n_t - (p_{m 2} 2b \ell/n_t) A(\omega) \exp[i((\omega t + q_0(\omega))].$$
where \( z_0 = - \ell b \Delta p / n_1 \), \( z_{1m}(\tau) = - p_m^* 2 \ell b (A_p f_p + B_p df_p / d\tau) \),

\[
A_p = \frac{n_1 - (m_i + M) \omega^2}{[n_1 - (m_i + M) \omega^2]^2 + [2K \omega]^2}, \quad B_p = - \frac{2K \omega}{[n_1 - (m_i + M) \omega^2]^2 + [2K \omega]^2},
\]

\( A(\omega) = 1/ \sqrt{[1 - (m_i + M) \omega^2 / n_1]^2 + [2K \omega / n_1]^2} \), \( \tan \varphi_1(\omega) = 2K \omega / ((m_i + M) \omega^2 - n_1) \).

It should be noted that \( A(\omega) \to 1 \) at \( \omega \to 0 \) and \( A(\omega) \to 0 \) at \( \omega \to \infty \).

Using the newly found expression for the wall shift (14), we can finally define the law of pressure change dynamics along the channel in a dimensional form

\[
p = p^+(\omega t) + \Delta p (1 - x/\ell) / 2 - p_m^*(1 - (x/\ell)^2) \Pi(\omega) \exp[i(\omega t + \varphi_p(\omega))] =
\]

\[
= \Delta p (1 - x/\ell) / 2 + p_m^* \exp[i(\omega t)] [1 - (1 - (x/\ell)^2) \Pi(\omega) \exp i \varphi_p(\omega)],
\]

where \( \Pi(\omega) = (3/2) M \omega^2 A(\omega) n_1^{-1} ((12 \gamma)^2 / (2 \varepsilon^2 \alpha^2) + 1)^{3/2} \),

\( \tan \varphi_p(\omega) = (2a B_p - 12 A_p) / (12 B_p + 2 \varepsilon^2 \alpha A) \).

It should be noted that \( \Pi(\omega) \to 0 \) at \( \omega \to 0 \) and \( \Pi(\omega) \to 3/2 \) at \( \omega \to \infty \).

The first component in the law of the channel wall shift (14) is the shift under static pressure difference \( \Delta p \), and the second component represents the shift, determined by the dynamic pressure in the channel. In the expression for pressure (15) the static component \( \Delta p (1 - x/\ell) / 2 \) reflects linear pressure fall along the channel, and the second component represents the pressure in the liquid due to its squeeze by elastically fixed channel wall.

In order to illustrate the application of the proposed mathematical model, the one-mass model with parameters: \( \ell = 0.1 \) m, \( h_0 / \ell = 1 / 20 \), \( \delta_0 / \ell = 1 / 15 \), \( \ell / b = 1 / 5 \), \( \rho = 1.84 \times 10^3 \) kg/m\(^3\), \( \nu = 2.5 \times 10^{-4} \) m\(^2\)/s, \( \mu = 0.3 \), \( \rho_0 = 7.87 \times 10^3 \) kg/m\(^3\), \( E = 1.96 \times 10^3 \) Pa are taken as example. The calculation results for \( A(\omega) \), \( \Pi(\omega) \) are presented on fig2.

![Fig. 2. Calculation results](image)

The solid line – the model with \( \kappa = 10^9 \) N/m\(^3\), the dash line 2 – the model with \( \kappa = 0 \).
Conclusions

The mathematical model on the basis of the one-mass model has been given for the vibration analysis of plate, interacting with pulsating viscous liquid layer, on Winkler foundations. The numerical examples show that model can be of use for evaluation the possibilities of resonance oscillations formation taking into account the frequency range of possible pressure pulsating at the channel butt end.

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