Two-Stage Sequential Procedure for the Estimation of the Powers of the Ratio of Two Exponential Scale Parameters

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Abstract

We consider the problem of minimum-risk point estimation of functions of the $r^{th}$ power of the ratio of two exponential scale parameters, $\theta = (\sigma_1/\sigma_2)^r$ where $r \neq 0$, under the assumption that each parameter exceeds some positive known number obtained from past experiments. Largo and Polestico [5] dealt on fully sequential procedure for the estimation of $\theta$. In this paper, a two-stage sequential scheme proposed by Duggan and Mukhopadhyay[1] on the given estimation problem is proposed. The procedure is shown to possess first-order properties and is risk efficient. Simulation results using $R$ language are provided.

Mathematics Subject Classification: 62L12

Keywords: exponential distributions, minimum-risk, risk efficiency, sequential analysis, simulation, stopping rule, two-stage
1 Introduction

Sequential analysis refers to a body of statistical theory and methods of analyzing data in which the final number of observations is not fixed in advance but is rather determined in the course of experiment by using some criteria as data become available, a contrary to the classical theory and practice which revolved around fixed-samples. In very general terms, sequential analysis is introduced to solve more efficiently a problem which has a fixed-sample solution and to deal with problems for which no fixed sample solution exists.

Sequential estimation of the scale parameter of exponential distributions has been extensively studied in the literature [3]. Under two-sample case, Uno [7] considered a sequential point estimation of the ratio of two exponential scale parameters, $\theta = \sigma_1/\sigma_2$, using fully sequential scheme. Futschik and Isogai [2] dealt with the generalization to any power of $\theta$, that is, $(\sigma_1/\sigma_2)^r$, for $r \neq 0$ and constructed a sequential confidence interval. Largo and Polestico [5] extended the work of Uno [7] to $r^{th}$ power of $\theta$. In this work, we consider a two-stage sequential procedure proposed by Duggan and Mukhopadhyay [1] in the estimation of the $r^{th}$ power of the ratio of two exponential scale parameters.

2 Minimum-Risk Sequential Point Estimation

Let $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ be independent observations from populations $\Pi_1$ and $\Pi_2$, respectively, where $\Pi_i$ is according to an exponential distribution with their corresponding densities as follows:

$$f_1(x) = \sigma_1^{-1} \exp(-x/\sigma_1)I(x > 0) \quad \text{and} \quad f_2(y) = \sigma_2^{-1} \exp(-y/\sigma_2)I(y > 0),$$

where $I(\cdot)$ stands for the usual indicator function of $(\cdot)$ and the scale parameters $\sigma_1 > 0$ and $\sigma_2 > 0$ are both unknown. Now, we want estimate the $r^{th}$ power of the ratio of two exponential scale parameters, $\theta = (\sigma_1/\sigma_2)^r$, $r \neq 0$. Taking samples of sizes $n$ and $m$ from $\Pi_1$ and $\Pi_2$, respectively, we then estimate $\theta = (\sigma_1/\sigma_2)^r$ by

$$\hat{\theta}_{(n,m)} = \left(\frac{X_n}{Y_m}\right)^r,$$

where $X_n$ and $Y_m$ are the natural estimators of the two scale parameters $\sigma_1$ and $\sigma_2$, respectively.

Given the random samples $X = (X_1, \ldots, X_n)$ and $Y = (Y_1, \ldots, Y_m)$, consider the quadratic loss function

$$L(\theta_{(n,m)}) = (\hat{\theta}_{(n,m)} - \theta)^2 + c(n + m),$$

where $c > 0$ is the known cost per observation from the two exponential
populations. Then, the risk as expected loss is given by

\[ R(n,m)(c) = E\{L(\hat{\theta}(n,m)) = E\left[\left(\left(\frac{X_n}{Y_m}\right)^r - \left(\frac{\sigma_1}{\sigma_2}\right)^r\right)^2\right] + c(n + m). \]

Now, Largo and Polestico [5] have shown that as \( n \to \infty \) and \( m \to \infty \),

\[ E\left[\left(\frac{X_n}{Y_m}\right)^r - \left(\frac{\sigma_1}{\sigma_2}\right)^r\right]^2 \approx r^2 \left(\frac{\sigma_1}{\sigma_2}\right)^{2r} \left(\frac{1}{n} + \frac{1}{m}\right). \]

Accordingly, for sufficiently large \( n \) and \( m \), the asymptotic risk is

\[ R_{n,m}(c) \approx r^2 \left(\frac{\sigma_1}{\sigma_2}\right)^{2r} \left(\frac{1}{n} + \frac{1}{m}\right) + c(n + m). \tag{1} \]

Then, taking random samples of equal size \( n = m \), the risk \( R_{n,m}(c) = R_n(c) \) is approximately minimized at

\[ n = m = rc^{-1/2} \left(\frac{\sigma_1}{\sigma_2}\right)^r = rc^{-1/2}\hat{\theta} \equiv n^*, \text{ say,} \tag{2} \]

with \( R_{n^*}(c) \approx 4cn^* \) for sufficiently small \( c \). But \( \sigma_1 \) and \( \sigma_2 \) are unknown, so the optimal sample size \( n^* \) needed to give the minimum risk, is expected to be unknown. The details of the nonexistence of fixed sample size procedures is found in Takada [6]. Thus, we develop a two-stage procedure for the estimation of \( \theta \) motivated by \( n^* \).

### 3 The Proposed Two-Stage Procedure

In a two-stage procedure, the choice of the first stage sample size \( m \) and the sequential procedure’s stopping rule \( N \) are motivated by the optimum sample size \( n^* \).

The first stage sample size is defined by

\[ m = \max \left\{ m_0, \left\lfloor \frac{r}{\sqrt{c}} \left(\frac{\sigma_L}{\sigma_G}\right)^r \right\rfloor + 1 \right\}, \tag{3} \]

where \( m_0 \geq 2 \) is a fixed integer and \( \lfloor x \rfloor \) is the floor function of \( x \). Based on the pilot samples \( X_1, ..., X_m \) and \( Y_1, ..., Y_m \), we calculate the corresponding sample means \( \overline{X}_m \) and \( \overline{Y}_m \) and define the stopping rule as

\[ N = \max \left\{ m, \left\lfloor \frac{r}{\sqrt{c}} \left(\frac{\overline{X}_m}{\overline{Y}_m}\right)^r \right\rfloor + 1 \right\}. \tag{4} \]
Here the stopping rule $N$ is the estimate of the unknown optimum sample size $n^*$ that minimizes the risk in (1). If $N = m$, there is no need to take any more samples at the second stage. Therefore, the final samples are $X_1, X_2, \ldots, X_m$ and $Y_1, Y_2, \ldots, Y_m$. If $N > m$, then one takes as the second sample the difference $N - m$ at the second stage. Thus, new observations $X_{m+1}, \ldots, X_N$ and $Y_{m+1}, \ldots, Y_N$ are added at the second stage. Thus, the final samples are $X_1, \ldots, X_m, X_{m+1}, \ldots, X_N$ and $Y_1, \ldots, Y_m, Y_{m+1}, \ldots, Y_N$. We then estimate $\theta = (\sigma_1/\sigma_2)^r$ by $\hat{\theta}_N = (\bar{X}_N/\bar{Y}_N)^r$ with associated regret defined by

$$R_N(c) - R_{n^*} = E\left(\hat{\theta}_N - \theta\right)^2 + 2cE(N) - 4cn^*.$$

The following theorems show the fundamental properties of the proposed two-stage procedure.

**Lemma 3.1** For the two-stage procedure defined in (3) and (4), as $c \to 0$

$$N \quad \frac{n^*}{a.s.} \to 1.$$

**Proof:** Let us consider two possible cases.

Case 1. If $N = m$, that is sampling stops at the first stage, then

$$N > \left\lfloor \frac{r}{\sqrt{c}} \left(\frac{\bar{X}_m/\bar{Y}_m}{\sigma_1/\sigma_2}\right)^r \right\rfloor + 1 > \frac{r}{\sqrt{c}} \left(\frac{\bar{X}_m/\bar{Y}_m}{\sigma_1/\sigma_2}\right)^r > 0.$$  

It follows that

$$\frac{r}{\sqrt{c}} \left(\frac{\bar{X}_m/\bar{Y}_m}{\sigma_1/\sigma_2}\right)^r \leq N \leq m + \frac{r}{\sqrt{c}} \left(\frac{\bar{X}_m/\bar{Y}_m}{\sigma_1/\sigma_2}\right)^r. \quad (5)$$

Dividing the whole inequality in (5) by $n^*$, we get

$$\frac{\left(\bar{X}_m, \bar{Y}_m\right)^r}{\left(\sigma_1/\sigma_2\right)^r} \leq \frac{N}{n^*} \leq \frac{m}{n^*} + \frac{\left(\bar{X}_m, \bar{Y}_m\right)^r}{\left(\sigma_1/\sigma_2\right)^r}.$$  

By the Strong Law of Large Numbers(SLLN), $\bar{X}_m \xrightarrow{a.s.} \sigma_1$ and $\bar{Y}_m \xrightarrow{a.s.} \sigma_2$ as $c \to 0$. Consequently, $(\bar{X}_m/\bar{Y}_m)^r \xrightarrow{a.s.} (\sigma_1/\sigma_2)^r$ as $c \to 0$. Thus,

$$\frac{N}{n^*} \geq \left(\frac{\bar{X}_m/\bar{Y}_m}{\sigma_1/\sigma_2}\right)^r \xrightarrow{a.s.} 1, \quad \text{as} \quad c \to 0.$$  

Now, since $n^* \xrightarrow{a.s.} \infty$ as $c \to 0$, for fixed $m$

$$\frac{N}{n^*} \leq \frac{m}{n^*} + \frac{\left(\bar{X}_m/\bar{Y}_m\right)^r}{\left(\sigma_1/\sigma_2\right)^r} \xrightarrow{a.s.} 1.$$
Two-stage procedure

Hence, $N/n^* \overset{a.s.}{\to} 1$.

Case 2. If $N > m$, that is sampling proceeds up to the second stage, then

$$N = \left[ \frac{r}{\sqrt{c}} \left( \frac{X_m}{Y_m} \right)^r \right] + 1 \leq \frac{r}{\sqrt{c}} \left( \frac{X_m}{Y_m} \right)^r + 1.$$

Thus,

$$\frac{r}{\sqrt{c}} \left( \frac{X_m}{Y_m} \right)^r \leq N \leq \frac{r}{\sqrt{c}} \left( \frac{X_m}{Y_m} \right)^r + 1. \quad (6)$$

Dividing the whole inequality in (6) by $n^*$, we have

$$\frac{(X_m/Y_m)^r}{(\sigma_1/\sigma_2)^r} \leq \frac{N}{n^*} \leq \frac{(X_m/Y_m)^r}{(\sigma_1/\sigma_2)^r} + \frac{1}{n^*}.$$

Using similar argument as in Case 1, we can show that

$$N/n^* > \frac{(X_m/Y_m)^r}{(\sigma_1/\sigma_2)^r} \overset{a.s.}{\to} 1, \quad \text{as} \quad c \to 0.$$

Now, since $n^* \overset{a.s.}{\to} \infty$ as $c \to 0$, it follows that

$$\frac{N}{n^*} \leq \frac{(X_m/Y_m)^r}{(\sigma_1/\sigma_2)^r} + \frac{1}{n^*} \overset{a.s.}{\to} 1.$$

Thus, $N/n^* \overset{a.s.}{\to} 1$. \hfill ■

The next theorem shows the first-order asymptotic efficiency of the two-stage procedure.

**Theorem 3.2** For the two-stage procedure defined in (3) and (4), we have

$$\lim_{c \to 0} \frac{E(N)}{n^*} = 1.$$

**Proof**: To prove the assertion, it suffices to show the following:

(a) $\lim_{c \to 0} \inf \frac{E(N)}{n^*} \geq 1,$

(b) $\lim_{c \to 0} \sup \frac{E(N)}{n^*} \leq 1.$

To show (a), by Fatou’s Lemma, we have

$$\liminf_{c \to 0} \frac{E(N)}{n^*} \geq E \left[ \liminf_{c \to 0} \frac{N}{n^*} \right].$$
However, by Lemma 3.1, \( N/n^* \xrightarrow{a.s.} 1 \) as \( c \to 0 \). Thus,

\[
\liminf_{c \to 0} \frac{E(N)}{n^*} \geq E\left[ \liminf_{c \to 0}(1) \right] = E(1) = 1.
\]

We are left to show that \( \lim_{c \to 0} \sup E(N) \leq 1 \) in (b). Now, from the stopping rule defined in (3) and (4), we have

\[
N - 1 \leq (m - 1)I(N = m) + \frac{r}{\sqrt{c}} \left( \frac{X_m}{Y_m} \right)^r
\]

\[
N \leq 1 + (m - 1)I(N = m) + \frac{r}{\sqrt{c}} \left( \frac{X_m}{Y_m} \right)^r. \tag{7}
\]

Taking the expectation on both sides of the inequality in (7), we get

\[
E(N) \leq 1 + (m - 1)P(N = m) + E\left[ \frac{r}{\sqrt{c}} \left( \frac{X_m}{Y_m} \right)^r \right].
\]

Observe that when sampling stops at the first stage, that is, if \( N = m \), then

\[
m \geq \left\lfloor \frac{r}{\sqrt{c}} \left( \frac{X_m}{Y_m} \right)^r \right\rfloor + 1 > \frac{r}{\sqrt{c}} \left( \frac{X_m}{Y_m} \right)^r.
\]

Hence,

\[
P(N = m) = P\left( m > \frac{r}{\sqrt{c}} \left( \frac{X_m}{Y_m} \right)^r \right),
\]

which implies that as \( c \to 0 \), \( P(N = m) \to P(m \geq \infty) = 0 \). It follows that

\[
\limsup_{c \to 0} \frac{E(N)}{n^*} \leq \frac{1}{n^*} + \frac{E\left[ \left( \frac{X_m}{Y_m} \right)^r \right]}{(\sigma_1/\sigma_2)^r}.
\]

Now, \( \left( \frac{X_m}{Y_m} \right)^r \xrightarrow{a.s.} (\sigma_1, \sigma_2)^r \) as \( X_m \xrightarrow{a.s.} \sigma_1 \) and \( Y_m \xrightarrow{a.s.} \sigma_2 \) whenever \( c \to 0 \). Hence, \( E\left[ \left( \frac{X_m}{Y_m} \right)^r \right] \xrightarrow{a.s.} (\sigma_1/\sigma_2)^r \). So, we have

\[
\limsup_{c \to 0} \frac{E(N)}{n^*} \leq \frac{1}{n^*} + 1.
\]

Moreover, \( \frac{1}{n^*} \xrightarrow{a.s.} 1 \), since \( n^* \to \infty \) as \( c \to 0 \). This shows that

\[
\limsup_{c \to 0} \frac{E(N)}{n^*} \leq 1.
\]
Accordingly,

$$1 \leq \liminf_{c \to 0} \frac{E(N)}{n^*} \leq \limsup_{c \to 0} \frac{E(N)}{n^*} \leq 1, \quad \text{as} \quad c \to 0.$$ 

Therefore, $$\lim_{c \to 0} \frac{E(N)}{n^*} = 1.$$ ■

Now, the performance of the sequential procedure is evaluated through the comparison of the two risks, namely $$R_N(c),$$ the risk associated by taking random samples of size $$N$$ resulting from the proposed sequential procedure, and the minimum risk $$R_{n^*}(c),$$ associated with the random samples of sizes equal to the optimal fixed sample size $$n^*.$$

**Theorem 3.3** For the two-stage procedure defined in (3) and (4), we have as $$c \to 0$$

$$\lim_{c \to 0} RE = \lim_{c \to 0} \frac{R_N(c)}{R_{n^*}(c)} = 1.$$ 

Here, we say that the proposed two-stage procedure is first-order asymptotically risk-efficient.

**Proof:** From (1) and (2), the risks associated with the stopping rule $$N,$$ is given by

$$R_N(c) = E \left[ \left( \frac{X_N}{Y_N} \right)^r - \left( \frac{\sigma_1}{\sigma_2} \right)^r \right] + 2cE(N). \quad (8)$$

Now, Lemma 1 of Isogai, et al.[4] gave the asymptotic distribution of functions of the exponential scale parameters. It follows that for $$\theta = (\sigma_1/\sigma_2)^r,$$

$$\sqrt{N} \left[ \left( \frac{X_N}{Y_N} \right)^r - \left( \frac{\sigma_1}{\sigma_2} \right)^r \right] \xrightarrow{D} N \left( 0, g(\sigma_1, \sigma_2) \right),$$

where $$g(\sigma_1, \sigma_2) = 2r^2 (\sigma_1/\sigma_2)^{2r},$$ and “$$\xrightarrow{D}$$” means convergence in distribution. Thus, as $$c \to 0$$

$$R_N(c) \approx 2r^2 \frac{1}{E(N)} \left( \frac{\sigma_1}{\sigma_2} \right)^{2r} + 2cE(N) = \frac{2cn^*}{E(N)} + 2cE(N),$$

and following directly from (2), $$R_{n^*}(c) \approx 4cn^*.$$ It follows from Theorem 3.2 that for sufficiently small $$c,$$

$$\frac{R_N(c)}{R_{n^*}(c)} \approx \frac{n^*}{2E(N)} + \frac{E(N)}{2n^*} \to 1.$$ ■
4 Simulation

In this section we shall present brief simulation results in order to verify the properties of the proposed two-stage procedure. We consider the case for \( r = 1 \) with \( \sigma_1 > \sigma_2 \) and \( \sigma_1 = \sigma_2 \) for \( r = 2 \). These correspond to the raw moments and are very useful in estimating the means and variances which are of great interest in inferential statistics. The two-stage procedure \( N \) defined by (3) and (4) was carried out with 10,000 independent replications for each sufficiently small \( c \) such that \( n^* = rc^{-1/2}\theta = 50, 200, 500, 800 \) and 1000. Also, we take \( \sigma_L \) and \( \sigma_G \) to be 50% and 30% away from \( \sigma_1 \) and \( \sigma_2 \), respectively.

The results of the simulation, based on the proposed two-stage procedure for the case for \( r = 1 \) and \( r = 2 \), are summarized in Tables 1 and 2, respectively, which contain the average estimate of \( \theta \), \( E(\hat{\theta}_N) \), the average stopping time, \( E(N) \), the ratio of \( E(N) \) and \( n^* \) and the risk-efficiency, \( RE(N) \).

Table 1: Simulation Results using the Proposed Two-stage Sequential Procedure with \( \theta = 1 \) where \( \sigma_1 = \sigma_2 = 3 \), \( \sigma_L = 1.5 \) and \( \sigma_G = 2.1 \) for \( r = 1 \) and fixed \( m_0 = 2 \).

<table>
<thead>
<tr>
<th>( n^* )</th>
<th>50</th>
<th>200</th>
<th>500</th>
<th>800</th>
<th>1000</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c )</td>
<td>0.0004</td>
<td>0.000025</td>
<td>0.000004</td>
<td>0.0000015</td>
<td>0.000001</td>
</tr>
<tr>
<td>( E(N) )</td>
<td>52.145</td>
<td>201.46</td>
<td>500.309</td>
<td>801.033</td>
<td>1000.934</td>
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<tr>
<td>( E(\hat{\theta}_N) )</td>
<td>0.9673674</td>
<td>0.990753</td>
<td>0.9966384</td>
<td>0.99974</td>
<td>0.9995965</td>
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<tr>
<td>( E(N)/n^* )</td>
<td>1.0429</td>
<td>1.0073</td>
<td>1.000618</td>
<td>1.0012912</td>
<td>1.0012912</td>
</tr>
<tr>
<td>( RE(N) )</td>
<td>1.0326326</td>
<td>1.009247</td>
<td>1.0033616</td>
<td>1.00026</td>
<td>1.0004035</td>
</tr>
</tbody>
</table>

Table 2: Simulation Results using the Proposed Two-stage Sequential Procedure with \( \theta = 1 \) where \( \sigma_1 = \sigma_2 = 3 \), \( \sigma_L = 1.5 \) and \( \sigma_G = 2.1 \) for \( r = 2 \) and fixed \( m_0 = 2 \).

<table>
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<tr>
<th>( n^* )</th>
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<th>200</th>
<th>500</th>
<th>800</th>
<th>1000</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c )</td>
<td>0.0016</td>
<td>0.0001</td>
<td>0.000016</td>
<td>0.0000062</td>
<td>0.000004</td>
</tr>
<tr>
<td>( E(N) )</td>
<td>59.682</td>
<td>209.734</td>
<td>509.415</td>
<td>813.881</td>
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<tr>
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<tr>
<td>( RE(N) )</td>
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<td>1.03001</td>
<td>1.036305</td>
<td>0.999775</td>
<td>0.999973</td>
</tr>
</tbody>
</table>

Observe that from Table 1 and Table 2, \( E(N) \) and \( E(\hat{\theta}_N) \) are, on the average, close to \( n^* \) and \( \theta \), respectively. Also, as \( c \to 0 \), \( E(N)/n^* \) and \( RE(N) \) each converges to 1 which confirm the first-order efficiency and first-order risk-efficiency for the proposed sequential procedure as given in Theorem 3.2 and Theorem 3.3, respectively. Thus, the proposed sequential procedure seems
to be useful and effective.

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