Pricing Default Bonds with Dynamic Default Barrier under the Hybrid Model

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Abstract

In this paper, we present a new pricing model for default bonds with dynamic default barrier under the hybrid model framework. Furthermore, closed-form pricing formulae for default bonds are obtained by using variable transforms and PDE approach, which expand the relevant literature’s results. Moreover, several numerical examples are provided to illustrate the effect of the main parameters on the credit spread of a default bond.

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1 Introduction

It is widely accepted that credit derivatives have become increasingly important in financial markets. Furthermore, most credit derivatives are sold in the form of bonds (especially corporate bonds). Therefore, the pricing problems of defaultable corporate bonds have received considerable attention in the last decade.

In the financial literature, the approaches used for pricing default bonds can be classified into three categories. The first group uses the structural bond-pricing approach started by Merton [1] and extended by Black and Cox [2],
Longstaff and Schwartz [3], Briys and de Varenne [4]. Within the framework of the structural approach, the default time is defined as the first crossing time of the firm’s value process through a default-triggering barrier. Such a default can be expected, but it is difficult to estimate the model’s parameters. The second group consists of models from the reduced-form approach, pioneered by Jarrow and Turnbull [5] and Duffie and Singleton [6]. In these models, the default is treated as an unpredictable event governed by a default intensity process. In this case, the default event can occur without any correlation with the firm’s value and such a default is called unexpected default. The drawback of the reduced-form approach is that it lacks the clear economic intuition. To combine the advantages of the two approaches, Madan and Unal [7] develop a new approach called the hybrid approach. Pricing default securities under the hybrid model are now a popular area of research. One is referred to Cathcart and El-Jahel [8], Realdon [9], Hyong-Chol O [10], Ballestra et al. [11]. In this paper, we will price a default bond with dynamic barrier under the hybrid model framework.

To cope with the possibility of early default before bond maturity, Black and Cox [2] assume a default-triggering level for the firm’s asset whereby default can occur at any time. This trigger level is introduced by considering a safety covenant that protects the bond-holder and given by an exponential function $K e^{-r(T-t)}$, where $K$ is the firm’s debt, $r$ is the short-term interest rate and it is a constant. Longstaff and Schwartz [3] extend the corporate bond pricing model of Black and Cox to allow the interest rate to follow the Vasicek model. However, they ignore the discount factor in default barrier and assume the default barrier is a constant $K$. Briys and de Varenne [4] develop a corporate bond valuation model that avoids some of the shortcomings of the previous literature. Especially, this model is characterized by a stochastic barrier $\alpha K P(r,t)$, where $P(r,t)$ denotes the price of a default-free zero-coupon bond according to the Hull-White interest rate model, $0 \leq \alpha \leq 1$ is a constant, which ensures that the bond-holder is not likely to get more payoff than the firm’s value in which case the bond defaults. Following the paper of Briys and de Varenne [4], Hui et al. [12] extend a corporate bond valuation model in which the default barrier is dynamic and it is given by $K P(r,t) e^{\beta_0 c_1(t)}$, where $\beta_0$ is a real number parameter to adjust the contribution of $c_1(t)$, and $c_1(t)$ is a real function which depends on the volatilities of the firm’s asset and the stochastic interest rate. Furthermore, a closed-form pricing formula is obtained by Lie algebra approach. Compared to the model of Hui et al. [12], our new model shows differences in several ways. On the one hand, we use the hybrid model instead of the structural model. On the other hand, we adopt PDE approach to derive a closed-form pricing formula.

The rest of the paper is organized as follows. In section 2, we provide some necessary assumptions and set up a pricing model for a default bond with
dynamic default barrier. In section 3, closed-form pricing formulae for default bonds are developed by using the variable transformation technique and PDE approach. In section 4, we discuss the influence of the main parameters on the bond’s credit spread. Conclusions are given in the last section.

2 Mathematical Model

In this section, we first introduce the basic assumptions. Some of them parallel those of Hui et al. [12] except the assumption of default intensity. Then, we build up a mathematical model for a default bond with dynamic default barrier.

2.1 Basic Assumptions

**Assumption 1.** The market is assumed to be arbitrage free, frictionless, complete and continuously open between 0 and \( T \). The stochastic information flow is described by a 3-dimensional Brownian motion \( W(t) = (W_V(t), W_r(t), W_\lambda(t)) \), defined on a complete probability space \( (\Omega, \mathcal{F}, Q) \) with a filtration \( \{\mathcal{F}_t, 0 \leq t \leq T\} \), where \( Q \) is the risk-neutral probability measure.

**Assumption 2.** The firm value (asset) \( V(t) \) follows a geometric Brownian motion:

\[
\frac{dV(t)}{V(t)} = r(t)dt + \sigma_1(t)dW_V(t), \quad V(0) = V_0,
\]

where \( \sigma_1(t) \) is a time-varying deterministic function and denotes the volatility of \( V(t) \), \( W_V(t) \) is a standard Brownian motion.

**Assumption 3.** The dynamics of the short-term interest rate \( r(t) \) obey the following stochastic differential equation:

\[
dr(t) = \left[ b_2(t) - a_2r(t) \right] dt + \sigma_2(t)dW_r(t), \quad r(0) = r_0,
\]

where \( a_2 > 0 \) is the speed rate of mean-reverting to a long-run mean \( \bar{b}_2(t) \), \( \sigma_2(t) \) is the volatility of \( r(t) \), \( W_r(t) \) is another standard Brownian motion, and it is dependent on \( W_V(t) \) with a correlation coefficient \( \rho_{12}(t) \), i.e., \( Cov(dW_V(t), dW_r(t)) = \rho_{12}(t)dt, |\rho_{12}(t)| < 1 \).

Under this assumption, the price \( P(r, t) \) at time \( t \) of a zero-coupon bond which can be considered default-free and pays 1 at time \( T > t \) is the solution of the following (see Wilmott [13]):

\[
\begin{align*}
\frac{\partial P}{\partial t} + \frac{1}{2}\sigma_2^2(t)\frac{\partial^2 P}{\partial r^2} + \left[ b_2(t) - a_2r \right] \frac{\partial P}{\partial r} - rP &= 0, \\
P(r, T) &= 1.
\end{align*}
\]

The solution of problem (3) is given by

\[
P(r, t) = e^{h_2(t) - h_1(t)r},
\]
where
\[ h_1(t) = \frac{1 - e^{a_2(t-T)}}{a_2}, h_2(t) = \int_t^T [h_1^2(v) \frac{\sigma_2^2(v)}{2} - b_2(v)h_1(v)]dv. \]

**Assumption 4.** The default happens with certainty when \( V(t) \) falls below a dynamic threshold level \( V_b(t) \), that is
\[ V_b(t) = KP(r,t)e^{\beta_0} \int_t^T [\sigma_1^2(s) + \sigma_2^2(s)h_1^2(s) + 2\rho_{12}(s)\sigma_1(s)\sigma_2(s)h_1(s)]ds, \]
and smaller than \( V_0 \), where \( K \) denotes the firm’s total debt and it is a given positive constant, \( \beta_0 \) denotes an adjusted factor, which is used to reflect the dynamic nature of boundary. Moreover, the unexpected default event can also happen as the first jump of a counting process whose intensity, denoted by \( \lambda(t) \), satisfies the following:
\[
d\lambda(t) = [b_3(t) - a_3\lambda(t)]dt + \sigma_3(t)dW_\lambda(t), \quad \lambda(0) = \lambda_0, \tag{5}
\]
where \( \lambda_0 \) and \( a_3 \) are positive constants, \( b_3(t) \) and \( \sigma_3(t) \) are time-varying deterministic functions. Moreover, \( W_\lambda(t) \) is a \( Q \)-standard Brownian motion and it is independent of \( W_V(t) \) and \( W_r(t) \).

**Remark 1.** It is well known that the Hull-White model can take negative values with very small probability in theory, which is by no means appealing from the financial viewpoint. Nevertheless, the use of Hull-White model is very popular in both academia and industry community since in this case one can obtain a closed-form solution of the securities (see e.g., Cathcart and El-Jahel [8], Realdon [9], Ballestra et al. [11]).

**Assumption 5.** Let \( B = B(V, r, \lambda, t) \) denote the price of a default bond with face value \( M \) at time \( T \). Following the idea of Briys and de Varenne [4], we suppose that (i) if the firm’s value hits the barrier \( V_b(t) \) during the life of the bond, the bond-holder receives a \( R \) default-free zero-coupon bond \( P(r,t) \), that is
\[
B(V, r, \lambda, t) = RP(r, t)M, \quad V \leq V_b(t), \tag{6}
\]
where \( R \) denotes a recovery rate; (ii) if the firm’s value has never breached the default barrier at time \( t < T \), but, it is not enough to pay the face value \( M \) at maturity \( T \), then the bond-holder can get the payoff \( \beta_1 V, 0 < \beta_1 \leq 1 \). Otherwise, its payoff is the face value \( M \), namely
\[
B(V, r, \lambda, T) = \begin{cases} 
M, & \text{if } V \geq M, \\
\beta_1 V, & \text{if } V < M,
\end{cases} \quad V > V_b(T). \tag{7}
\]
2.2 Pricing problem

In this subsection, we will set up a pricing model for a default bond by using Ito’s lemma and the standard arbitrage-free principle. Construct a portfolio:

\[ \Pi = B - \Delta P, \]  

where \( \Delta \) can be chosen so that \( \Pi \) is risk-free in the interval \( (t, t + dt) \), that is,

\[ d\Pi = r\Pi dt. \]  

If the bond does not default in the interval \( (t, t+dt) \), then applying Ito’s lemma to Equation (8), we obtain

\[ d\Pi^{(1)}_t = dB - \Delta dP \]

\[ = \frac{\partial B}{\partial t} dt + \frac{\partial B}{\partial V} dV + \frac{\partial B}{\partial r} dr + \frac{\partial B}{\partial \lambda} d\lambda + \frac{1}{2} \frac{\partial^2 B}{\partial V^2} (dV)^2 + \frac{1}{2} \frac{\partial^2 B}{\partial r^2} (dr)^2 \]

\[ + \Delta \left( \frac{\partial P}{\partial t} dt + \frac{\partial P}{\partial r} dr + \frac{1}{2} \frac{\partial^2 P}{\partial r^2} (dr)^2 \right). \]

To eliminate the stochastic noise of the above equation, we can choose \( \Delta = \frac{\partial B}{\partial r} \).

On the other hand, if the bond defaults in the time interval, with a probability of \( \lambda dt \), then the change in the value of the portfolio is

\[ d\Pi^{(2)}_t = -B + G + O(\sqrt{dt}), \]  

This is due to the sudden loss of the risky bond and the holder of the bond can receive \( G \) for making up for the loss. Moreover, \( O(\sqrt{dt}) \) denotes higher order infinitesimal of \( dt \). Taking expectation in \( d\Pi_t \) and neglecting higher order infinitesimal terms of \( dt \), we have

\[ E(d\Pi_t) = (\lambda dt)E(d\Pi^{(2)}_t) + (1 - \lambda dt)E(d\Pi^{(1)}_t) \approx (\lambda dt)E(d\Pi^{(2)}_t) + E(d\Pi^{(1)}_t). \]  

Substituting \( d\Pi^{(1)}_t \), \( (\lambda dt) \), \( (1 - \lambda dt) \), \( E(d\Pi^{(2)}_t) \), \( E(d\Pi^{(1)}_t) \) into (9), and after some calculations, we have

\[ \frac{\partial B}{\partial t} + \frac{1}{2} \left[ \sigma_1(t)^2 V^2 \frac{\partial^2 B}{\partial V^2} + \sigma_2(t) \frac{\partial^2 B}{\partial V^2} + \sigma_3(t) \frac{\partial^2 B}{\partial V^2} + 2 \rho_{12}(t) \sigma_1(t) \sigma_2(t) V \frac{\partial^2 B}{\partial V \partial V} \right] \]

\[ + [b_2(t) - a_2] \frac{\partial B}{\partial r} + [b_3(t) - a_3 \lambda] \frac{\partial B}{\partial \lambda} + \rho_1 V \frac{\partial B}{\partial V} = (r + \lambda) B - \lambda G. \]  

According to the assumption 5, the price of the default bond with dynamic barrier satisfies the following problem in the domain \( \Sigma_1 = \{(V, r, \lambda, t) : V_b(t) < V, -\infty < r < +\infty, -\infty < \lambda < +\infty, 0 < t < T \} \):

\[ \begin{cases} \frac{\partial B}{\partial t} + \frac{1}{2} \left[ \sigma_1(t)^2 V^2 \frac{\partial^2 B}{\partial V^2} + \sigma_2(t) \frac{\partial^2 B}{\partial V^2} + \sigma_3(t) \frac{\partial^2 B}{\partial V^2} + 2 \rho_{12}(t) \sigma_1(t) \sigma_2(t) V \frac{\partial^2 B}{\partial V \partial V} \right] \\ + [b_2(t) - a_2] \frac{\partial B}{\partial r} + [b_3(t) - a_3 \lambda] \frac{\partial B}{\partial \lambda} + \rho_1 V \frac{\partial B}{\partial V} = (r + \lambda) B - \lambda RP(r, t) M, \\ B(V_b(r, t), r, \lambda, t) = R P(r, t) M, \quad 0 \leq t \leq T, \\ B(V, r, \lambda, T) = \begin{cases} M, & \text{if } V \geq M, \\ \beta_1 V, & \text{if } V < M, \end{cases} \\ B(V_b(T)) = V_b(T) < V. \end{cases} \]
3 Closed-form Pricing Formulae for Default Bonds

In this section, we adopt the variable transformation technique and PDE approach to derive a closed-form solution of problem (13). Note that (13) is a complicated boundary value problem. Thus, we should first transform this problem into a simpler PDE problem by changing numeraire, namely, let

\[ V = yP(r, t), \ B(V, r, \lambda, t) = u(y, \lambda, t)P(r, t)M. \]  
(14)

Substituting the above equations into (13), and after some calculations, we have

\[
\begin{aligned}
\frac{\partial u}{\partial t} + \frac{y^2}{2} \sigma^2(t) \frac{\partial^2 u}{\partial y^2} + \frac{\sigma^2(t) \sigma_2^2(t)}{2} \frac{\partial^2 u}{\partial \lambda^2} + \left[ b_3(t) - a_3 \lambda \right] \frac{\partial u}{\partial \lambda} - \lambda u &= -R \lambda, \\
u(y_b(t), \lambda, t) &= R, \\
0 &\leq t \leq T,
\end{aligned}
\]  
(15)

where

\[
\sigma^2(t) = \sigma_1^2(t) + \sigma_2^2(t)h_1^2(t) + 2\rho_1(t)\sigma_1(t)\sigma_2(t)h_1(t),
\]  
(16)

\[
y_b(t) = Ke^{b_3(0)T} \sigma^2(s)ds.
\]  
(17)

To solve problem (15), we make the following transformations

\[
\begin{aligned}
x &= \log \frac{y}{y_b(t)}, \\
\theta &= \lambda e^{a_3(t-T)} + c(t), \\
u(y, \lambda, t) &= \phi(x, \theta, t)e^{\int_t^T \gamma(s)ds+A(t)\lambda} + R,
\end{aligned}
\]  
(18)

where

\[
A(t) = \frac{e^{a_3(t-T)} - 1}{a_3}, \gamma(t) = A(t)b_3(t) + A^2(t)\frac{\sigma_2^2(t)}{2}, c(t) = \int_t^T \left[ b_3(s) + A(s)\sigma_3^2(s) \right] e^{a_3(s-T)} ds.
\]

Substituting (18) into (15) yields the following problem:

\[
\begin{aligned}
\frac{\partial \phi}{\partial t} + \frac{\sigma^2(t)}{2} \frac{\partial^2 \phi}{\partial x^2} + \frac{\sigma^2(t) \sigma_2^2(t)}{2} \frac{\partial^2 \phi}{\partial \theta^2} - \frac{(1-2\beta_0)\sigma^2(t)}{2} \frac{\partial \phi}{\partial x} &= 0, \\
\phi(0, \theta, t) &= 0, \\
0 &\leq t \leq T, \\
\phi(x, \theta, T) &= \begin{cases} 
1 - R, & \text{if } x \geq \log \frac{M}{K} \\
\beta_1 Ke^x - R, & \text{if } 0 < x < \log \frac{M}{K},
\end{cases}
\end{aligned}
\]  
(19)

In what follows, we adopt the PDE method of images (see Buchen [14]) to solve problem (19) in the domain \( \Sigma_2 = \{(x, \theta, t) : 0 < x < +\infty, -\infty < \theta < +\infty, 0 < t < T\} \). Therefore, we define

\[
F(x) = \begin{cases} 
f(x), & \text{if } x > 0, \\
0, & \text{if } x < 0,
\end{cases}
\]  
(20)
where
\[ f(x) = \begin{cases} 1 - R, & \text{if } x \geq \log \frac{M}{K}, \\ \frac{\beta_1 K}{M} e^x - R, & \text{if } 0 < x < \log \frac{M}{K}. \end{cases} \]

By the image method, we may solve the following problem in the domain \( \Sigma_3 = \{(x, \theta, t) : -\infty < x < +\infty, -\infty < \theta < +\infty, 0 < t < T\} \), that is
\[
\begin{align*}
\frac{\partial \phi_1}{\partial t} + \frac{\sigma^2(t)}{2} \frac{\partial^2 \phi_1}{\partial x^2} + \frac{\sigma^2(t)e^{2a_3(t-T)}}{2} \frac{\partial^2 \phi_1}{\partial \theta^2} - \frac{(1-2\beta_0)\sigma^2(t)}{2} \frac{\partial \phi_1}{\partial x} &= 0, \\
\phi_1(x, \theta, T) &= F(x),
\end{align*}
\]
and then restricting the solution in the domain \( \Sigma_2 \). Applying Fourier transforms (see Jiang [15]) to problem (21), we have
\[
\begin{align*}
\frac{d\varphi}{dt} - \varphi \left[ \frac{\sigma^2(t)}{2} \xi^2 + \frac{\sigma^2(t)e^{2a_3(t-T)}}{2} \eta^2 + i\xi \frac{(1-2\beta_0)\sigma^2(t)}{2} \right] &= 0, \\
\varphi(\xi, \eta, T) &= F(\xi),
\end{align*}
\]
where \( \varphi(\xi, \eta, t) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \phi_1(x, \theta, t)e^{-i(\xi x + \eta \theta)}d\theta dx, F(\xi) = \int_{-\infty}^{+\infty} F(x)e^{-ix}dx. \)

Moreover, by the method of separation of variables, the solution to problem (22) is given by
\[
\varphi(\xi, \eta, t) = F(\xi)e^{-[d_1 \xi^2 + d_3 \eta^2 + i\xi d_2]},
\]
where
\[
d_1 = \int_{t}^{T} \frac{\sigma^2(s)}{2} ds, d_2 = \int_{t}^{T} \frac{(1-2\beta_0)\sigma^2(s)}{2} ds, d_3 = \int_{t}^{T} \frac{\sigma^2(s)e^{2a_3(s-T)}}{2} ds.
\]

Performing the inverse Fourier transform to (23), we obtain
\[
\phi_1(x, \theta, t) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} F(\xi)G(x - \xi, \theta - \eta)d\xi d\eta,
\]
where
\[
G(x, \theta) = \frac{e^{-\frac{[d_1 \xi^2 + d_3 \eta^2 + 2a_3(s-T)]}{4d_1d_3}}}{4\pi \sqrt{d_1d_3}}.
\]

Substituting (20) and (25) into (24), and after tedious calculations, we have
\[
\phi_1(x, \theta, t) = \frac{\beta_1 K}{M} e^{d_1 \xi - d_2 \eta} \left[ N\left(\frac{-2d_1 \xi - d_2 \eta + \log \frac{M}{K}}{\sqrt{2d_1d_3}}\right) - N\left(\frac{-2d_1 \xi - d_2 \eta}{\sqrt{2d_1d_3}}\right) - RN\left(\frac{x-d_2 \eta}{\sqrt{2d_1d_3}}\right) + N\left(\frac{x-d_2 \eta - \log \frac{M}{K}}{\sqrt{2d_1d_3}}\right)\right],
\]
where \( N(v) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{v} e^{-\frac{s^2}{2}} ds \) is the cumulative probability distribution function of a standard normal distribution.
To obtain the solution to problem (19), we let \( \phi_2(x, \theta, t) = \phi_1(-x, \theta, t) \), and then \( \phi_2(0, \theta, t) = \phi_1(0, \theta, t) \). Thus, \( \phi(x, \theta, t) = \phi_1(x, \theta, t) - \phi_2(x, \theta, t) \) is the solution of problem (19). Furthermore, by Equation (26), we have

\[
\phi(x, \theta, t) = \frac{\beta_1 K}{M} e^{d_1 - d_2 + x} \left[ N\left(\frac{-2d_1 - x + d_2 + \log \frac{M}{K}}{\sqrt{2d_1}}\right) - N\left(\frac{-2d_1 - x + d_2}{\sqrt{2d_1}}\right)\right]
- \frac{\beta_1 K}{M} e^{d_1 - d_2 - x} \left[ N\left(\frac{-2d_1 + x + d_2 + \log \frac{M}{K}}{\sqrt{2d_1}}\right) - N\left(\frac{-2d_1 + x + d_2}{\sqrt{2d_1}}\right)\right].
\]

By the above transforms, we can have the following result, namely

**Theorem 1.** The pricing formula of a default bond with dynamic default barrier under the hybrid model is given by

\[
B(V, r, \lambda, t) = MP(r, t)(R + \left(\frac{\beta_1 V}{MP(r, t)}\right) [N(-\sqrt{\int_t^T \sigma^2(s)ds} - d_1^* + \frac{\log \frac{M}{K}}{\sqrt{\int_t^T \sigma^2(s)ds}}) - N(-d_1^* - \sqrt{\int_t^T \sigma^2(s)ds})] + \frac{\beta_1 K^2 P(r, t)}{MV} e^{2\beta_0 \int_t^T \sigma^2(s)ds} [N(-\sqrt{\int_t^T \sigma^2(s)ds} + d_2^* + \frac{\log \frac{M}{K}}{\sqrt{\int_t^T \sigma^2(s)ds}}) - N(-d_2^* - \sqrt{\int_t^T \sigma^2(s)ds})]\),
\]

where

\[
\int_t^T \gamma(s)ds = \frac{1}{a_3} \int_t^T b_3(s)[e^{a_3(s-T)} - 1]ds + \frac{1}{2a_3^2} \int_t^T \sigma_3^2(s)[e^{2a_3(s-T)} - 2e^{a_3(s-T)} + 1]ds,
\]

\[
\int_t^T \sigma^2(s)ds = \int_t^T \sigma_1^2(s)ds + \frac{2}{a_3^2} \int_t^T \rho_{12}(s)\sigma_1(s)\sigma_2(s)[1 - e^{a_2(s-T)}]ds + \frac{1}{a_3^2} \int_t^T \sigma_2^2(s)[e^{2a_2(s-T)} - 2e^{a_2(s-T)} + 1]ds,
\]

\[
d_1^* = \frac{\log \frac{V}{KP(r, t)} - \frac{\int_t^T \sigma^2(s)ds}{2}}{\sqrt{\int_t^T \sigma^2(s)ds}},
\]

According to **Theorem 1** and letting \( a_3 = 0, b_3(t) = 0, \sigma_3(t) = 0 \), we can derive the pricing formula for a default bond with dynamic default barrier under the structural model, namely

**Theorem 2.** The pricing formula of a default bond with dynamic default
barrier under the structural model is given by

\[
B(V, r, t) = MP(r, t)(R + \{ \frac{\beta_1 V}{MP(r, t)}[N(-\sqrt{\int_t^T \sigma^2(s)ds} - d_1^* + \frac{\log \frac{M}{K}}{\sqrt{\int_t^T \sigma^2(s)ds}}) - N(-d_1^* - \sqrt{\int_t^T \sigma^2(s)ds} - d_2^* + \frac{\log \frac{M}{K}}{\sqrt{\int_t^T \sigma^2(s)ds}})]

- \frac{\beta_1 K^2 P(r, t) e^{2\beta_0 \int_t^T \sigma^2(s)ds}}{MV} [N(-\sqrt{\int_t^T \sigma^2(s)ds} + d_2^* + \frac{\log \frac{M}{K}}{\sqrt{\int_t^T \sigma^2(s)ds}}) - N(-d_2^* - \sqrt{\int_t^T \sigma^2(s)ds} - d_2^* + \frac{\log \frac{M}{K}}{\sqrt{\int_t^T \sigma^2(s)ds}})\}].
\]

(29)

**Remark 2.** The Theorems 2 is just the result obtained by Hui et al. [12].

### 4 Numerical Analysis

In the bond market, a very important problem is how to quantify the difference between default bonds and risk-free bonds (usually treasury bonds). Following the paper of Hui et al. [12], we introduce the term of credit spreads. The credit spread is defined as the difference between the yields of a default bond and its corresponding non-defaultable bond. Based on the above pricing formulae, we have the following formula of credit spread at time \( t \)

\[
\text{Credit Spread} = \begin{cases} 
- \frac{1}{T-t} \log \frac{B(V, r, \lambda, t)}{P(r, t)}, & \text{Hybrid model,} \\
- \frac{1}{T-t} \log \frac{B(V, r, t)}{P(r, t)}, & \text{Structural model.}
\end{cases}
\]

(30)

For analytical convenience, but without loss of generality, we suppose that all the parameters, such as \( \sigma_1(t), \sigma_2(t), \sigma_3(t), b_2(t), b_3(t), \) and \( \rho_{12}(t) \) are constants over time \( t \in [0, T] \). Moreover, throughout this section, unless otherwise stated, the related parameters are as follows:

\[
V = 10.5, K = 1, \sigma_1 = 0.25, r = 0.05, a_2 = 1.0, b_2 = 0.05, \sigma_2 = 0.0316, \\
\rho_{12} = -0.25, a_3 = 0.01, b_3 = 0.0136, \sigma_3 = 0.0237, \lambda = 0.00291, \beta_1 = 1, \\
\beta_0 = 0, T = 10, R = 0.48, M = 1.
\]

where the above parameters are taken from the papers of Ballestra and Pacelli [11] and Hui et al. [12], and then used to show that our model is different from the traditional model.
Figure 1: Default barriers for different $\beta_0$

Figure 2: Credit spreads of structural model for different $\beta_0$

Figure 3: Credit spreads of hybrid model for different $\beta_0$
Figure 4: Credit spreads for hybrid model and structural model

From the above figures, we can see that the credit spread of structural model is significantly higher than that of hybrid model, and the credit spread of the hybrid model is not sensitive to the parameter $\beta_0$. We postulate that the default intensity considered reduces the credit spread.

5 Conclusion

In this paper, we present a new pricing model for a default bond with dynamic default barrier based on the hybrid model. Furthermore, a closed-form solution is obtained by the variable transforms and the image approach as well as Fourier transforms, which extend the relevant literature’s results. Moreover, we provide some numerical examples to highlight our model. Numerical results show that (i) by adjusting the parameter $\beta_0$, we can get different default barriers, and then obtain the corresponding credit spreads; (ii) the credit spread of structural model is significantly higher than that of hybrid model.

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References


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