The Uncertainty Volatility Models and Tree Approximation

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Abstract

We consider the “Knightian uncertainty” volatility models as the optimal stochastic control problem. The main goal is the approach of the tree approximate algorithm for the calculation of the upper and lower prices of the European option. For this we use the theory of the viscosity solution of the $G$-heat equation.

Keywords: stochastical volatility, uncertainty volatility, $G$-heat equation, viscosity solution, stochastical optimal control, tree approximation

Introduction

The fundamental works of Black-Scholes [3] have made the significant contribution to the option-pricing theory. The numerous statistical supervision have shown that the Black Scholes model is inconsistent with the market prices, because the market volatility is not constant as in the Black-Scholes model. This problem is known as “the volatility smile”. The majority of financial researchers focuses on stochastic volatility models. In these models the volatility is the stochastic process. Stochastic volatility models were first studied by Johnson and Shanno [7], Hull and White [6], Scott [11]. Other models for the volatility dynamics were proposed by E. Stein and J. Stein [12], Heston [5]. In these models

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the dynamic of asset prices was driven volatility process that may or may not be independent. The may be most popular model was introduced by Henston [5].

The last time the uncertainty volatility models are involving attention of researchers. The uncertainty volatility models are closed stochastic volatility models. The scientific work on uncertainty volatility and superhedging were papers by M. Avellaneda, A. Levy, and A. Par’as [1], and G. Meyer [8]. In these models the uncertainty volatility is the family of deterministic trajectories. The theory of G-expectation by Peng [9] is directly connected with uncertainty volatility models, but in this paper we will consider the problem of the superhedging as the stochastic control problem and viscosity solution of Bellman-Hamilton-Jacoby equation will be used.

The stochastic volatility model

Let \( \exp(-rt)S_t \) be the discount price of an asset at time \( t \). The Black-Scholes risk-neutral equation for the asset price is:

\[
dS_t = S_t \left( r dt + \sigma dW_t \right), \ln S_0 = x.
\]

The risk-neutral equation for the discount price is:

\[
d\left( \exp(-rt)S_t \right) = \sigma \left( \exp(-rt)S_t \right) dW_t, \ln S_0 = x.
\]

Here the process \( W \) is Brownian motion. The European option price with pay off \( f(x) \) in the terminal time \( T \) is

\[
V(x,0) = \exp(-rT)Ef(S_T) = \frac{\exp(-rt)}{\sqrt{2\pi \sigma^2 T}} \int_{-\infty}^{\infty} f \left( x + \left( r - \frac{\sigma^2}{2} \right) T + y \right) \exp \left( -\frac{y^2}{2\sigma^2 T} \right) dy.
\]

We introduce the function \( \varphi(x) = f(\exp(x)) \). Let the function \( \nu(t,x) \) be

\[
\nu(x,t) = \exp(-r(T-t))Ef(S_T)/\ln S_t = x = \exp(-r(T-t))E(\varphi(\ln S_T)/\ln S_t = x) = \exp(-r(T-t))E(\varphi(Y_T)|Y_t = x) = \frac{\exp(-r(T-t))}{\sqrt{2\pi \sigma^2 (T-t)}} \int_{-\infty}^{\infty} \varphi \left( x + \left( r - \frac{\sigma^2}{2} \right) (T-t) + y \right) \exp \left( -\frac{y^2}{2\sigma^2 (T-t)} \right) dy.
\]

The process \( Y \) is: \( Y_t = \ln S_t \), and \( dY_t = \left( r - \frac{\sigma^2}{2} \right) dt + \sigma dW_t \). The function \( \nu \) is the solution of the heat equation:

\[
\frac{\partial \nu}{\partial t} + \left( r - \frac{\sigma^2}{2} \right) \frac{\partial \nu}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2 \nu}{\partial x^2} - rv = 0,
\]

with terminal condition \( \nu(x,T) = \varphi(x) \). The price \( V(x,0) = \exp(-rT)\nu(x,0) \). It is well known that the Cox-Ross-Rubinstein binary model approaches the Black-Scholes model and tree algorithm is used for the evaluation of the option price.
Farwell we assume that the volatility \( \sigma \) is the stochastic process, which is dependent or is not dependent from the asset price. For example, the Henston model is the system of stochastic differential equations for risk-neutral asset price:

\[
S_t = \exp(Y_t), \quad dY_t = \left( r - \frac{U_t}{2} \right) dt + \sqrt{U_t} dW_t, \quad dU_t = k(\theta - U_t) dt + \xi \sqrt{U_t} dZ_t, \quad Y_0 = \ln S_0, U_0 = \sigma_0,
\]

Here the process \( Z \) is Brownian motion, and \( E(dW_t dZ_t) = \rho dt \).

Let function \( v \) be \( v(x, y, t) = E(\varphi(Y_T)|Y_t = x, U_t = y) \). The function \( v \) is the solution of the two-dimensional heat equation:

\[
\frac{\partial v}{\partial t} + \left( r - \frac{y}{2} \right) \frac{\partial v}{\partial x} + k(\theta - y) \frac{\partial v}{\partial y} + \frac{y}{2} \frac{\partial^2 v}{\partial x^2} + \rho \xi y \frac{\partial^2 v}{\partial x \partial y} + \frac{\xi^2 y}{2} \frac{\partial^2 v}{\partial y^2} = 0
\]

with terminal condition \( v(x, y, T) = \varphi(x) \). The option price is:

\[
V(\ln S_0, \sigma_0, 0) = \exp(-rT) v(x, y, 0).
\]

It is not well known about the binary model, which would approach the Henston’s model. To construct such model let’s consider the discreet time equations:

\[
\Delta Y^n_{tm} = \left( r - \frac{U^n_{tm+1}}{2} \right) \frac{T}{n} + \sqrt{U^n_{tm+1}} \frac{T}{n} \varepsilon_{m}, \quad \Delta U^n_t = k(\theta - U^n_{tm}) + \xi \sqrt{U^n_{tm}} \frac{T}{n} \zeta_{m}, \quad Y^n_0 = \ln S_0, U^n_0 = \sigma_0,
\]

\[
t_k = \frac{T}{n}, \quad k = 0, \ldots, n.
\]

The sequences \( \varepsilon, \zeta \) are i.i.d. sequences and \( E \varepsilon_i = \nu \varepsilon_i^2 = \nu \), \( E \zeta_i \zeta_j = \left\{ \begin{array}{cl} \rho, & i = j \\ 0, & i \neq j \end{array} \right. \)

Let’s introduce the function

\[
v^n(x, y, t_{m-1}) = E(\varphi(Y^n_T)|Y^n_{tm-1} = x, U^n_{tm-1} = y),
\]

and for this functions the next recurrent formulas are correct:

\[
v^n(x, y, t_{m-1}) = E(v^n(x + \Delta Y^n_t, y + \Delta U^n_t, t)|Y^n_{tm-1} = x, U^n_{tm-1} = y), v^n(x, y, T) = \varphi(x).
\]

The next theorem is true.

**Theorem 1.** If the function \( \varphi(x) \) is continuous and bounded then

\[
\lim_{n \to \infty} v^n(x, y, 0) = v(x, y, 0).
\]

From this theorem follows that model (7) approach the Henston’s model (5).
The two uncertainty volatility models

In this part we will consider the risk-neutral model of asset price:

$$S_t = \exp(Y_t), dY_t = \left( r - \frac{\sigma_t^2}{2} \right) dt + \sigma_t dW_t, Y_0 = \ln S_0.$$  \hspace{1cm} (9)

Unlike the last in the equation (9) the function $\sigma_t$ is the uncertainty volatility, which satisfies to inequalities:

$$\sigma_t \leq \sigma_t \leq \bar{\sigma}_t.$$ \hspace{1cm} (10)

In inequalities (10) the functions $\sigma_t$ and $\bar{\sigma}_t$ are deterministic functions. Let the set $\Delta_{t_1}^{t_2}$ be $\Delta_{t_1}^{t_2} = \{ \sigma_t : \sigma_t \leq \sigma_t \leq \bar{\sigma}_t, t \in [t_1, t_2] \}$. To evaluate upper and lower prices we must to solve the next stochastic control problems:

$$\sup_{\sigma \in \Delta_{t_1}^{t_2}} E_{\sigma} \phi(Y_T); \inf_{\sigma \in \Delta_{t_1}^{t_2}} E_{\sigma} \phi(Y_T).$$ \hspace{1cm} (11)

There are the function $\bar{v}(x,t) = \sup_{\sigma \in \Delta_{t_1}^{t_2}} E(\phi(Y_T)|Y_t = x)$ and the function $v(x,t) = \inf_{\sigma \in \Delta_{t_1}^{t_2}} E(\phi(Y_T)|Y_t = x)$, then the upper price is:

$$\bar{V} = \exp(-rT)\bar{v}(\ln S_0, 0)$$

and the lower price is:

$$V = \exp(-rT)v(\ln(S_0), 0)$$

The Hamilton-Jacoby-Bellman (HJB) equations for functions $\bar{v}(x,t)$ and $v(x,t)$ are:

$$\frac{\partial \bar{v}}{\partial t} + r \frac{\partial \bar{v}}{\partial x} + \sup_{\sigma \leq \sigma \leq \bar{\sigma}_t} \frac{\sigma^2}{2} \left( \frac{\partial^2 \bar{v}}{\partial x^2} - \frac{\partial \bar{v}}{\partial x} \right) = 0,$$ \hspace{1cm} (12)

$$\frac{\partial v}{\partial t} + r \frac{\partial v}{\partial x} + \inf_{\sigma \leq \sigma \leq \bar{\sigma}_t} \frac{\sigma^2}{2} \left( \frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x} \right) = 0,$$

with terminal condition: $v(T,x) = \phi(x)$. These equations are equivalent next equations:

$$\frac{\partial \bar{v}}{\partial t} + (r-A_t) \frac{\partial \bar{v}}{\partial x} + A_t \frac{\partial^2 \bar{v}}{\partial x^2} + B_t \left| \frac{\partial^2 \bar{v}}{\partial x^2} - \frac{\partial \bar{v}}{\partial x} \right| = 0,$$

$$\frac{\partial v}{\partial t} + (r-A_t) \frac{\partial v}{\partial x} + A_t \frac{\partial^2 v}{\partial x^2} + B_t \left| \frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x} \right| = 0; A_t = \bar{\sigma}_t^2 - \sigma_t^2, B_t = \frac{\sigma_t^2 - \sigma_t^2}{2}.$$ \hspace{1cm} (13)

The equations (13) are G-heat equations with nonlinear operators:

$$G_1(t, \frac{\partial \bar{v}}{\partial x}, \frac{\partial^2 \bar{v}}{\partial x^2}) = A_t \left( \frac{\partial^2 \bar{v}}{\partial x^2} - \frac{\partial \bar{v}}{\partial x} \right), G_2(t, \frac{\partial v}{\partial x}, \frac{\partial^2 v}{\partial x^2}) = A_t \left( \frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x} \right) - B_t \left| \frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x} \right|.$$
If \( \varphi(x) \) is continuous and bounded function, \( A_t \) and \( B_t \) are continuous functions on interval \([0,T]\), equations (13) have unique viscosity solutions [13]. The Perron’s verification of the viscosity solutions (13) as solutions of stochastic control problems (11) is standard [2]. In financial literature equations (13) are known as Barenblot equations [8], when \( A_t \) and \( B_t \) are constants.

In the first model let’s \( \sigma_t = \sigma_0 \exp(\delta t) \), and there is uncertainly parameter \( \delta \in [\delta, \delta] \). Then \( A_t = \frac{\sigma_0(\exp(\delta t) + \exp(-\delta t))}{2}, B_t = \frac{\sigma_0(\exp(\delta t) - \exp(-\delta t))}{2} \).

In symmetric case: \( \delta \in [-\alpha, \alpha] \) there are equations (13):

\[
\begin{align*}
\frac{\partial \varphi}{\partial t} + (r - \sigma_0ch(at))\frac{\partial \varphi}{\partial x} + \sigma_0ch(at)\frac{\partial^2 \varphi}{\partial x^2} + \sigma_0sh(at)\left(\frac{\partial^2 \varphi}{\partial x^2} - \frac{\partial \varphi}{\partial x}\right) &= 0, \\
\frac{\partial \psi}{\partial t} + (r - \sigma_0ch(at))\frac{\partial \psi}{\partial x} + \sigma_0ch(at)\frac{\partial^2 \psi}{\partial x^2} - \sigma_0sh(at)\left(\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial \psi}{\partial x}\right) &= 0.
\end{align*}
\]

If \( \varphi(x) \) – terminal condition – is unbounded as the European option call: \( \varphi(x) = \max \{\exp(x) - K, 0\} \), then it is necessary to apply the next trick. Let’s \( \bar{\psi} = \exp(\alpha x) \overline{\psi}, u = \exp(\alpha x) \psi \) and \( \exp(-\alpha x) \varphi(x) \) is bounded and continuous function. From (14) next equations for functions \( \bar{\psi}, u \) follow:

\[
\begin{align*}
\frac{\partial \bar{\psi}}{\partial t} + (r - A_t)\left(\frac{\partial \bar{\psi}}{\partial x} - \alpha\right)\bar{u} + A_t \left(\frac{\partial \bar{\psi}}{\partial x} - \alpha\right)^2 \bar{u} + B_t \left(\frac{\partial \bar{\psi}}{\partial x} - \alpha\right)^2 \bar{u} - \left(\frac{\partial \bar{\psi}}{\partial x} - \alpha\right)\bar{u} &= 0, \\
\frac{\partial u}{\partial t} + (r - A_t)\left(\frac{\partial u}{\partial x} - \alpha\right)u + A_t \left(\frac{\partial u}{\partial x} - \alpha\right)^2 u - B_t \left(\frac{\partial u}{\partial x} - \alpha\right)^2 u - \left(\frac{\partial u}{\partial x} - \alpha\right)u &= 0,
\end{align*}
\]

with terminal conditions: \( \bar{u}(x,T) = u(x,T) = \exp(-rx) f(x) \). Equations (15) are the \( G \)-heat equations with bounded, continuous terminal conditions and therefore have unique viscosity solutions.

In the second model let’s uncertainty volatility:

\( d\sigma_t = \delta \sigma_t dt \) (16)

In the equation (16) \( \delta_t \in [\delta, \delta] \). For the second model we must solve the next problems:

\[
\begin{align*}
\sup_{\delta \leq \delta \leq \delta} E_{\delta \varphi(Y_T)}; \inf_{\delta \leq \delta \leq \delta} E_{\delta \varphi(Y_T)}, \ dY_t = \left(r - \frac{\sigma_t^2}{2}\right)dt + \sigma_t dW_t, Y_0 = \ln S_0 \quad \text{and} \\
\quad d\sigma_t = \delta \sigma_t dt, \quad \text{with } \sigma_0 \text{- initial condition.}
\end{align*}
\]

The HJB equations for problems (17) is
\[
\frac{\partial \nu}{\partial t} + r \frac{\partial \nu}{\partial x} + \frac{y^2}{2} \left( \frac{\partial^2 \nu}{\partial x^2} - \frac{\partial \nu}{\partial y} \right) + A_y \frac{\partial \nu}{\partial y} + yB_t \frac{\partial \nu}{\partial y} = 0, \quad (18)
\]

with terminal conditions \( v(x, y, t) = \varphi(x) \).

The equations (18) are two dimensional \( G \)-heat equations with
\[
G_1 \left( y, t, \frac{\partial \nu}{\partial x}, \frac{\partial \nu}{\partial y}, \frac{\partial^2 \nu}{\partial x^2} \right) = r \frac{\partial \nu}{\partial x} + \frac{y^2}{2} \left( \frac{\partial^2 \nu}{\partial x^2} - \frac{\partial \nu}{\partial y} \right) + A_y \frac{\partial \nu}{\partial y} + yB_t \frac{\partial \nu}{\partial y} \]
and \( G_2 \left( y, t, \frac{\partial \nu}{\partial x}, \frac{\partial \nu}{\partial y}, \frac{\partial^2 \nu}{\partial x^2} \right) = r \frac{\partial \nu}{\partial x} + \frac{y^2}{2} \left( \frac{\partial^2 \nu}{\partial x^2} - \frac{\partial \nu}{\partial y} \right) + A_y \frac{\partial \nu}{\partial y} - yB_t \frac{\partial \nu}{\partial y} \).

In the symmetrical case: \( \bar{\sigma}_t = \alpha_t, \bar{\delta}_t = -\alpha_t \), the equations (18) will look as follows:
\[
\frac{\partial \nu}{\partial t} + r \frac{\partial \nu}{\partial x} + \frac{y^2}{2} \left( \frac{\partial^2 \nu}{\partial x^2} - \frac{\partial \nu}{\partial y} \right) + y\alpha_t \frac{\partial \nu}{\partial y} = 0, \\
\frac{\partial \nu}{\partial t} + r \frac{\partial \nu}{\partial x} + \frac{y^2}{2} \left( \frac{\partial^2 \nu}{\partial x^2} - \frac{\partial \nu}{\partial y} \right) - y\alpha_t \frac{\partial \nu}{\partial y} = 0.
\]

The tree approximation uncertainty volatility models

There is the well known result, what the Cox-Ross-Rubinstein model approximates the Black-Scholes model. In this part we introduce binary models, which approximate the uncertainly volatility models from the above part.

Let’s consider the sequence of points on the interval \([0, T] \): \( t_k = \frac{kT}{n} \), with the i.i.d. sequence \( \varepsilon \): \( E\varepsilon_i = 0, \ E\varepsilon_i^2 = 1 \), the filtration \( F_i = \sigma(\varepsilon_1, \ldots, \varepsilon_i) \) and the sequence
\[
Y_i^\sigma = Y_{i-1}^\sigma + T \left( r - \frac{\sigma^2}{2} \right) + \sigma_i \sqrt{\frac{T}{n}} \varepsilon_i, Y_0^\sigma = 0. \]

Let’s consider following standard problems of the stochastic discrete optimum control:
\[
\sup_{\sigma \in \Delta^n_m} E \varphi(x + Y_n^\sigma), \ \inf_{\sigma \in \Delta^n_m} E \varphi(x + Y_n^\sigma). \quad (19)
\]

There is in (19) the set \( \Delta^n_m = \left\{ (\sigma_i)^n_m : \sigma_i = \sigma_i \leq \bar{\sigma}_i \right\} \). We assume that for any \( x \) \( E \varphi(x + Y_n^\sigma) \) is bounded. It allows for the problem decision to use functions:
\[
\bar{v}^n(x, t_{m-1}) = \sup_{\sigma \in \Delta^n_m} E \left( \varphi(Y_n^\sigma) \right) \big| Y_{i-1}^\sigma = x \) and \( v^n(x, t_{m-1}) = \inf_{\sigma \in \Delta^n_m} E \left( \varphi(Y_n^\sigma) \right) \big| Y_{i-1}^\sigma = x \).
As the sequence $\varepsilon$ is i.i.d., for this functions are correct equalities:

$$
\bar{v}^n (x, t_{m-1}) = \sup_{\sigma \in \Delta_m} E\varphi \left( x + \sum_{i=m}^{n} \frac{T}{n} \left( r - \frac{\sigma_i^2}{2} \right) + \sigma_i \sqrt{\frac{T}{n}} \varepsilon_i \right) \\
\underline{v}^n (x, t_{m-1}) = \sup_{\sigma \in \Delta_m} E\varphi \left( x + \sum_{i=m}^{n} \frac{T}{n} \left( r - \frac{\sigma_i^2}{2} \right) + \sigma_i \sqrt{\frac{T}{n}} \varepsilon_i \right),
$$

From this it follows, that

$$
\bar{v}^n (x, t_{m-1}) = \sup_{\sigma_m \leq \sigma \leq \bar{\sigma}_m} \int \sup_{\sigma \in \Delta_m} \varphi \left( x + \frac{T}{n} \left( r - \frac{\sigma_m^2}{2} \right) + \sigma_m \sqrt{\frac{T}{n}} \varepsilon_i \right) P_\varepsilon (dy) = \\
\inf_{\sigma_m \leq \sigma \leq \bar{\sigma}_m} \int \bar{v}^n \left( x + \frac{T}{n} \left( r - \frac{\sigma_m^2}{2} \right) + \sigma_m \sqrt{\frac{T}{n}} \varepsilon_i \right) P_\varepsilon (dy).
$$

Therefore the next recurrent formulas are correct.

$$
\sup_{\sigma \leq \sigma_m \leq \sigma} \left[ E\bar{v} \left( x + \frac{T}{n} \left( r - \frac{\sigma_m^2}{2} \right) + \sigma_m \sqrt{\frac{T}{n}} \varepsilon_i \right) \right] = 0,
$$

$$
\inf_{\sigma \leq \sigma_m \leq \sigma} \left[ E\underline{v} \left( x + \frac{T}{n} \left( r - \frac{\sigma_m^2}{2} \right) + \sigma_m \sqrt{\frac{T}{n}} \varepsilon_i \right) \right] = 0, m = 1, 2, ..., n + 1,
$$

and

$$
\bar{v}^n (x, T) = \underline{v}^n (x, T) = \varphi (x),
$$

$$
\bar{v}^n (x, 0) = \sup_{\sigma \in \sigma^0} E\varphi \left( x + \sum_{i=1}^{n} \frac{T}{n} \left( r - \frac{\sigma_i^2}{2} \right) + \sigma_i \sqrt{\frac{T}{n}} \varepsilon_i \right),
$$

$$
\underline{v}^n (x, 0) = \inf_{\sigma \in \sigma^0} E\varphi \left( x + \sum_{i=1}^{n} \frac{T}{n} \left( r - \frac{\sigma_i^2}{2} \right) + \sigma_i \sqrt{\frac{T}{n}} \varepsilon_i \right). \text{ If we can apply the Taylor's formula, we will obtain the differential equations (13).}
$$

Thus, the next theorem is true.

**Theorem 2.** If the function $\varphi (x)$ is continuous and bounded, then

$$
\bar{v} (x, 0) = \lim_{n \to \infty} \sup_{\sigma \in \sigma^0} E\varphi \left( x + \sum_{i=1}^{n} \frac{T}{n} \left( r - \frac{\sigma_i^2}{2} \right) + \sigma_i \sqrt{\frac{T}{n}} \varepsilon_i \right) = \lim_{n \to \infty} \bar{v}^n (x, 0),
$$

$$
\underline{v} (x, 0) = \lim_{n \to \infty} \inf_{\sigma \in \sigma^0} E\varphi \left( x + \sum_{i=1}^{n} \frac{T}{n} \left( r - \frac{\sigma_i^2}{2} \right) + \sigma_i \sqrt{\frac{T}{n}} \varepsilon_i \right) = \lim_{n \to \infty} \underline{v}^n (x, 0).
$$
There are in (20) $\overline{v}(x,t)$, $\underline{v}(x,t)$ viscosity solutions of equations (13) with terminal conditions: $\overline{v}(x,T) = \underline{v}(x,T) = \varphi(x)$, and $\sigma_i = \sigma, \sigma_t = \sigma$.

Let’s define $P(\varepsilon_i = -1) = P(\varepsilon_i = 1) = \frac{1}{2}$ and will obtain tree approximation of the first model from the above part.

For the tree approximation of the second model of the uncertainly volatility let’s consider control discrete problems:

\[
\sup_{\delta \in \Delta^n_i} E\varphi \left( x + \sum_{i=1}^{n} \left( \frac{T}{n} \left( r - \frac{y^2}{2} \prod_{j=1}^{i} \left( 1 + \frac{T \delta_j}{n} \right)^2 \right) + \sqrt{T/n} \prod_{j=1}^{i} \left( 1 + \frac{T \delta_j}{n} \right) \varepsilon_i \right) \right),
\]

\[
\inf_{\delta \in \Delta^n_i} E\varphi \left( x + \sum_{i=1}^{n} \left( \frac{T}{n} \left( r - \frac{y^2}{2} \prod_{j=m}^{i} \left( 1 + \frac{T \delta_j}{n} \right)^2 \right) + \sqrt{T/n} \prod_{j=m}^{i} \left( 1 + \frac{T \delta_j}{n} \right) \varepsilon_i \right) \right).
\]

There is the set $\Delta^n_i = \left\{ (\delta_i)_{m} : \delta \leq \delta_i \leq \delta^\top \right\}$. Let’s consider functions

\[
\overline{v}^n(x, y, t_{m-1}) = \sup_{\sigma \in \Delta^n_m} E\varphi \left( x + \sum_{i=m}^{n} \left( \frac{T}{n} \left( r - \frac{y^2}{2} \prod_{j=m}^{i} \left( 1 + \frac{T \delta_j}{n} \right)^2 \right) + \sqrt{T/n} \prod_{j=m}^{i} \left( 1 + \frac{T \delta_j}{n} \right) \varepsilon_i \right) \right)
\]

and

\[
\underline{v}^n(x, t_{m-1}) = \inf_{\sigma \in \Delta^n_m} E\varphi \left( x + \sum_{i=m}^{n} \left( \frac{T}{n} \left( r - \frac{y^2}{2} \prod_{j=m}^{i} \left( 1 + \frac{T \delta_j}{n} \right)^2 \right) + \sqrt{T/n} \prod_{j=m}^{i} \left( 1 + \frac{T \delta_j}{n} \right) \varepsilon_i \right) \right).
\]

Thus the similar result for the second model is the Theorem 3.

**Theorem 3.** If the function $\varphi(x)$ is continuous and bounded, then

\[
\overline{v}(x, y, 0) = \lim_{n \to \infty} \sup_{\delta \in \Delta^n_i} E\varphi \left( x + \sum_{i=1}^{n} \left( \frac{T}{n} \left( r - \frac{y^2}{2} \prod_{j=1}^{i} \left( 1 + \frac{T \delta_j}{n} \right)^2 \right) + \sqrt{T/n} \prod_{j=1}^{i} \left( 1 + \frac{T \delta_j}{n} \right) \varepsilon_i \right) \right) = \frac{1}{2},
\]

\[
\lim_{n \to \infty} \overline{v}^n(x, y, 0) = \frac{1}{2},
\]

\[
\underline{v}(x, y, 0) = \lim_{n \to \infty} \inf_{\delta \in \Delta^n_i} E\varphi \left( x + \sum_{i=1}^{n} \left( \frac{T}{n} \left( r - \frac{y^2}{2} \prod_{j=1}^{i} \left( 1 + \frac{T \delta_j}{n} \right)^2 \right) + \sqrt{T/n} \prod_{j=1}^{i} \left( 1 + \frac{T \delta_j}{n} \right) \varepsilon_i \right) \right) = \frac{1}{2},
\]

\[
\lim_{n \to \infty} \underline{v}^n(x, y, 0) = \frac{1}{2}.
\]
In (22) there are $v(x,t)\), $v(x,t)$ viscosity solutions of equations (18) with terminal conditions: $v(x,y,T) = v(x,y,T) = \phi(x)$, and $\delta^{-}\delta^{-} = \delta^{-} \delta^{-}$.

The proof of the theorem 1 directly follows from the Taylor’s formula, the proofs of theorems 2, 3 are based on the technique of half-relaxed limits from the theory of approximation schemes for fully nonlinear partial differential equations and are standardized; therefore we do not present these proofs here, see, for example, [10].

**Conclusion**

The basic result consists, that it possible to replace the normal distribution to the binary distribution in the Euler’s scheme for the stochastic differential equation. Thanks to it, for the first uncertainty volatility model, for example, we can construct the next tree calculation algorithm:

1. $v^{n}(x,T) = v^{n}(x,T) = \phi(x)$;

2. $v^{n}(x,t_{m-1}) = v^{n}(x,t_{m}) + \frac{T}{2n_{h}} \left( v^{n}(x+h,t_{m}) - v^{n}(x-h,t_{m}) \right) +$

\[
\frac{T}{2n_{h}} \sup_{\sigma_{n} \leq \sigma_{m} \leq \sigma_{n}} \left\{ \sigma^{2} \left( \frac{v^{n}(x+h,t_{m}) - 2v^{n}(x,t_{m}) + v^{n}(x-h,t_{m})}{h} - \frac{v^{n}(x+h,t_{m}) - v^{n}(x-h,t_{m})}{2} \right) \right\}.
\]

\[
v^{n}(x,t_{m-1}) = v^{n}(x,t_{m}) + \frac{T}{2n_{h}} \left( v^{n}(x+h,t_{m}) - v^{n}(x-h,t_{m}) \right) +
\]

\[
\frac{T}{2n_{h}} \inf_{\sigma_{n} \leq \sigma_{m} \leq \sigma_{n}} \left\{ \sigma^{2} \left( \frac{v^{n}(x+h,t_{m}) - 2v^{n}(x,t_{m}) + v^{n}(x-h,t_{m})}{h} - \frac{v^{n}(x+h,t_{m}) - v^{n}(x-h,t_{m})}{2} \right) \right\}.
\]

$x \in X_{m-1}$,

$X_{0} = \{x_{0}\}$, $X_{k} = \{x_{0} - kh, x_{0} - (k-1)h, ..., x_{0}-x_{0} + (k-1)h, x_{0} + (k-1)h\}$

Here $h$ is the step of the partition of the phase scale. This algorithm is economical and fast, see [4].

**References**


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