Geometry of Lightlike Hypersurfaces of an Indefinite Trans-Sasakian Manifold with Quarter-Symmetric Metric Connection

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Abstract

Recently, paper [6] studied lightlike hypersurfaces of an indefinite trans-Sasakian manifold with a quarter-symmetric metric connection. In this paper, we study further the geometry of this subject. The object of this paper is to study two types of lightlike hypersurfaces, named by recurrent and Lie recurrent lightlike hypersurfaces, of an indefinite trans-Sasakian manifold with a quarter-symmetric metric connection.

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1 Introduction

The notion of quarter-symmetric metric connection on a Riemannian manifold was introduced by K. Yano and T. Imai [9]. This definition on indefinite trans-Sasakian manifold was presented by D.H. Jin [6]. We quote Jin’s definition as follow: An affine connection $\nabla$ on a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is said to be a quarter-symmetric connection if its torsion tensor $\bar{T}$ satisfies

$$\bar{T}(X, Y) = \theta(Y)JX - \theta(X)JY,$$  \hspace{1cm} (1.1)
for any vector fields $\bar{X}$ and $\bar{Y}$ on $\bar{M}$, where $J$ is a $(1,1)$-type tensor field and $\theta$ is a 1-form on $\bar{M}$. Moreover, if this connection $\bar{\nabla}$ is satisfied $\bar{\nabla}\bar{g} = 0$, then it is called a \textit{quarter-symmetric metric connection}.

The notion of trans-Sasakian manifold of type $(\alpha, \beta)$ was introduced by J.A. Oubina [8]. Indefinite Sasakian, Kenmotsu and cosymplectic manifolds are important kinds of indefinite trans-Sasakian manifold such that

$$\alpha = 1, \quad \beta = 0; \quad \alpha = 0, \quad \beta = 1; \quad \alpha = \beta = 0,$$

respectively.

The purpose of this paper is to study two types of lightlike hypersurfaces of an indefinite trans-Sasakian manifold $(\bar{M}, J, \zeta, \theta, \bar{g})$ with a quarter-symmetric metric connection, which are called \textit{recurrent} and \textit{Lie recurrent} lightlike hypersurfaces of $\bar{M}$. We assume that the tensor field $J$ and the 1-form $\theta$, defined by (1.1), are identical with the structure tensor field $J$ and the structure 1-form $\theta$ of the indefinite trans-Sasakian structure $(J, \zeta, \theta, \bar{g})$ on $\bar{M}$, respectively.

Denote by $\bar{\nabla}$ the Levi-Civita connection of $\bar{M} \equiv (\bar{M}, J, \zeta, \theta, \bar{g})$ with respect to the metric $\bar{g}$. We define a linear connection $\bar{\nabla}$ on $\bar{M}$ given by

$$\bar{\nabla}_X\bar{Y} = \bar{\nabla}_X\bar{Y} - \theta(X)J\bar{Y}. \quad (1.2)$$

Due to [7], it is known that a linear connection $\bar{\nabla}$ on $\bar{M}$ is a quarter-symmetric metric connection if and only if $\bar{\nabla}$ satisfies (1.2). In this paper, we study the geometry of lightlike hypersurfaces $\bar{M}$ of an indefinite trans-Sasakian manifold $\bar{M}$ with a quarter-symmetric metric connection $\bar{\nabla}$ given by (1.2).

2 Preliminaries

An odd-dimensional semi-Riemannian manifold $(\bar{M}, \bar{g})$ is called an \textit{indefinite almost contact metric manifold} if there exist a $(1,1)$-type tensor field $J$, a vector field $\zeta$, called the \textit{structure vector field}, and a 1-form $\theta$ such that

$$J^2\bar{X} = -\bar{X} + \theta(\bar{X})\zeta, \quad \bar{g}(J\bar{X}, J\bar{Y}) = \bar{g}(\bar{X}, \bar{Y}) - \epsilon\theta(\bar{X})\theta(\bar{Y}), \quad \theta(\zeta) = 1, \quad (2.1)$$

for any vector fields $\bar{X}$ and $\bar{Y}$ on $\bar{M}$, where $\epsilon = 1$ or $-1$ according as $\zeta$ is spacelike or timelike, respectively. The set $\{J, \zeta, \theta, \bar{g}\}$ is called an \textit{indefinite almost contact metric structure} of $\bar{M}$. From (2.1), we show that

$$J\zeta = 0, \quad \theta \circ J = 0, \quad \theta(\bar{X}) = \epsilon\bar{g}(\bar{X}, \zeta), \quad \bar{g}(J\bar{X}, \bar{Y}) = -\bar{g}(\bar{X}, J\bar{Y}).$$

In the entire discussion of this article, we shall assume that the structure vector field $\zeta$ is a spacelike one, \textit{i.e.}, $\epsilon = 1$, without loss of generality.

**Definition.** An indefinite almost contact metric manifold $(\bar{M}, \bar{g})$ is said to be an \textit{indefinite trans-Sasakian manifold} if there exist a Levi-Civita connection $\bar{\nabla}$ and two smooth functions $\alpha$ and $\beta$ such that

$$(\bar{\nabla}_X J)\bar{Y} = \alpha\{\bar{g}(\bar{X}, \bar{Y})\zeta - \theta(\bar{Y})\bar{X}\} + \beta\{\bar{g}(J\bar{X}, \bar{Y})\zeta - \theta(\bar{Y})J\bar{X}\}. $$
We say that \( \{J, \zeta, \theta, \bar{g}\} \) is an indefinite trans-Sasakian structure of type \((\alpha, \beta)\).

Replacing the Levi-Civita connection \( \tilde{\nabla} \) by the quarter-symmetric metric connection \( \bar{\nabla} \), the equation in the above definition is reformed to

\[
(\bar{\nabla}_X J)\bar{Y} = (\tilde{\nabla}_X J)\bar{Y} = \alpha\{\bar{g}(\bar{X}, \bar{Y})\zeta - \theta(\bar{Y})\bar{X}\} + \beta\{\bar{g}(J\bar{X}, \bar{Y})\zeta - \theta(\bar{Y})J\bar{X}\}.
\]

Replacing \( Y \) by \( \zeta \) to (2.2) and using \( J\zeta = 0 \) and \( \theta(\bar{\nabla}_X \zeta) = 0 \), we obtain

\[
\bar{\nabla}_X \zeta = -\alpha JX + \beta (X - \theta(X)\zeta).
\]

Let \((M, g)\) be a lightlike hypersurface of \( \bar{M} \). Denote by \( F(M) \) the algebra of smooth functions on \( M \) and by \( \Gamma(E) \) the \( F(M) \) module of smooth sections of a vector bundle \( E \) over \( M \). Also denote by \( (2.1)_i \) the \( i \)-th equation of the three equations in (2.1). We use same notations for any others. It is known [4] that the normal bundle \( TM^\perp \) of \( M \) is a subbundle of the tangent bundle \( TM \), of rank 1, and coincides with the radical distribution \( \text{Rad}(TM) = TM \cap TM^\perp \).

A complementary vector bundle \( S(TM) \) of \( TM^\perp \) in \( TM \) is non-degenerate distribution on \( M \), which is called a screen distribution on \( M \), such that

\[
TM = TM^\perp \oplus_{\text{orth}} S(TM),
\]

where \( \oplus_{\text{orth}} \) denotes the orthogonal direct sum. For any null section \( \xi \) of \( TM^\perp \) on a coordinate neighborhood \( U \subset M \), there exists a unique null section \( N \) of a unique vector bundle \( \text{tr}(TM) \) in \( S(TM)^\perp \) satisfying

\[
\bar{g}(\xi, N) = 1, \quad \bar{g}(N, N) = \bar{g}(N, X) = 0, \quad \forall X \in \Gamma(S(TM)).
\]

We call \( \text{tr}(TM) \) and \( N \) the transversal vector bundle and the null transversal vector field of \( M \) with respect to the screen distribution \( S(TM) \) respectively. The tangent bundle \( T\bar{M} \) of \( \bar{M} \) is decomposed as follow:

\[
T\bar{M} = TM \oplus \text{tr}(TM) = \{TM^\perp \oplus \text{tr}(TM)\} \oplus_{\text{orth}} S(TM).
\]

In the sequel, let \( X, Y, Z \) and \( W \) be the vector fields on \( M \), unless otherwise specified. Let \( P \) be the projection morphism of \( TM \) on \( S(TM) \). Then the local Gauss and Weingartan formulas of \( M \) and \( S(TM) \) are given respectively by

\[
\begin{align*}
\bar{\nabla}_X Y &= \nabla_X Y + B(X, Y)N, \quad (2.4) \\
\bar{\nabla}_X N &= -A_X X + \tau(X)N, \quad (2.5) \\
\bar{\nabla}_X PY &= \nabla_X PY + C(X, PY)\xi, \quad (2.6) \\
\bar{\nabla}_X \xi &= -A_\xi \xi - \sigma(X)\xi, \quad (2.7)
\end{align*}
\]

where \( \nabla \) and \( \nabla^* \) are the induced linear connections on \( M \) and \( S(TM) \) respectively, \( B \) and \( C \) are the local second fundamental forms on \( M \) and \( S(TM) \) respectively.
respectively, $A_\eta$ and $A_\xi^*$ are the shape operators on $M$ and $S(TM)$ respectively, and $\tau$ and $\sigma$ are 1-forms on $M$.

It is known [5] that, for any lightlike hypersurface $M$ of an indefinite almost contact metric manifold $\tilde{M}$, $J(TM^\perp)$ and $J(tr(TM))$ are subbundles of $S(TM)$, of rank 1. In the following, we shall assume that $\zeta$ is tangent to $M$. Călin [2] proved that if $\zeta$ is tangent to $M$, then it belongs to $S(TM)$. In this case, there exists two non-degenerate almost complex distributions $D_o$ and $D$ with respect to $J$, i.e., $J(D_o) = D_o$ and $J(D) = D$, such that

$$S(TM) = J(TM^\perp) \oplus J(tr(TM)) \oplus_{orth} D_o,$$

$$D = TM^\perp \oplus_{orth} J(TM^\perp) \oplus_{orth} D_o,$$

$$TM = D \oplus J(tr(TM)).$$

Consider two null vector fields $U$ and $V$ and two 1-forms $u$ and $v$ such that

$$U = -JN, \quad V = -J\xi, \quad u(X) = g(X, V), \quad v(X) = g(X, U). \quad (2.8)$$

Denote by $S$ the projection morphism of $TM$ on $D$. Any vector field $X$ of $M$ is expressed as $X = SX + u(X)U$. Applying $J$ to this form, we have

$$JX = FX + u(X)N, \quad (2.9)$$

where $F$ is a tensor field of type $(1,1)$ globally defined on $M$ by $FX = JSX$. Applying $J$ to (2.9) and using (2.1) and (2.8), we have

$$F^2X = -X + u(X)U + \theta(X)\zeta. \quad (2.10)$$

Denote by $\tilde{R}$, $R$ and $R^*$ the curvature tensors of the quarter-symmetric metric connection $\nabla$ on $\tilde{M}$, and the induced linear connections $\nabla$ and $\nabla^*$ on $M$ and $S(TM)$ respectively. Using the Gauss-Weingarten formulas, we obtain two Gauss-Codazzi equations for $M$ and $S(TM)$ such that

$$\tilde{R}(X, Y)Z = R(X, Y)Z + B(X, Z)A_\eta Y - B(Y, Z)A_\eta X \quad (2.11)$$

$$+ \{ (\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) + \tau(X)B(Y, Z) - \tau(Y)B(X, Z) + B(T(X, Y), Z) \} N,$$

$$\tilde{R}(X, Y)N = -\nabla_X (A_\eta Y) + \nabla_Y (A_\eta X) + A_\eta [X, Y] + \tau(X)A_\eta Y - \tau(Y)A_\eta X$$

$$+ \{ B(Y, A_\eta X) - B(X, A_\eta Y) + 2\sigma(X, Y) \} N,$$

$$R(X, Y)PZ = R^*(X, Y)PZ + C(X, PZ)A_\xi Y - C(Y, PZ)A_\xi X \quad (2.12)$$

$$+ \{ (\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ) - \sigma(X)C(Y, PZ) + \sigma(Y)C(X, PZ) + C(T(X, Y), PZ) \} \xi,$$

$$R(X, Y)\xi = -\nabla^*_X (A_\xi Y) + \nabla^*_Y (A_\xi X) + A_\xi [X, Y] - \sigma(X)A_\xi Y + \sigma(Y)A_\xi X$$

$$+ \{ C(Y, A_\xi^* X) - C(X, A_\xi^* Y) - 2\sigma(X, Y) \} \xi. \quad (2.13)$$
Let $(\bar{M}, \bar{g})$ be an indefinite trans-Sasakian manifold with a quarter-symmetric metric connection $\bar{\nabla}$. Using (1.1), (2.4) and (2.9), we obtain
\[
(\bar{\nabla}_X \bar{g})(Y,Z) = B(X,Y)\eta(Z) + B(X,Z)\eta(Y), \tag{3.1}
\]
\[
T(X,Y) = \theta(Y)FX - \theta(X)FY, \tag{3.2}
\]
\[
B(X,Y) - B(Y,X) = \theta(Y)u(X) - \theta(X)u(Y), \tag{3.3}
\]
where $T$ is the torsion tensor with respect to $\nabla$ and $\eta$ is a 1-form such that $\eta(X) = \bar{g}(X,N)$.

From the fact that $B(X,Y) = \bar{g}(\bar{\nabla}_X Y, \xi)$, we know that $B$ is independent of the choice of the screen distribution $S(TM)$ and satisfies
\[
B(X,\xi) = B(\xi, X) = 0. \tag{3.4}
\]
The local second fundamental forms are related to their shape operators by
\[
B(X,Y) = g(A^*_X X, Y), \quad \bar{g}(A^*_X X, N) = 0, \tag{3.5}
\]
\[
C(X,PY) = g(A_X X, PY), \quad \bar{g}(A_X X, N) = 0. \tag{3.6}
\]
Applying $\bar{\nabla}_X$ to $\bar{g}(N,\xi)$ and using (2.4), (2.6) and (3.4), we get $\tau = \sigma$ and
\[
\bar{\nabla}_X \xi = -A^*_X X - \tau(X)\xi. \tag{3.7}
\]
Taking $X = \xi$ to (3.5) and using (3.4) and $S(TM)$ is non-degenerate, we get
\[
A^*_X = 0. \tag{3.8}
\]
Substituting (2.9) into (2.3) and using (2.4), we have
\[
\nabla_X \xi = -\alpha FX + \beta\{X - \theta(X)\xi\}, \quad B(X,\xi) = -\alpha u(X). \tag{3.9}
\]
Applying $\nabla_X$ to $\bar{g}(\zeta, N) = 0$ and using (2.3), (2.5) and (3.6), we have
\[
C(X,\zeta) = -\alpha v(X) + \beta\eta(X). \tag{3.10}
\]
Applying $\nabla_X$ to (2.8) and (2.9), we have
\[
B(X,U) = C(X,V) \equiv \rho(X), \tag{3.11}
\]
\[
\nabla_X U = F(A_X X) + \tau(X)U - \{\alpha\eta(X) + \beta v(X)\}\zeta, \tag{3.12}
\]
\[
\nabla_X V = F(A^*_X X) - \tau(X)V - \beta u(X)\zeta, \tag{3.13}
\]
\[
(\nabla_X F)(Y) = u(Y)A_X X - B(X,Y)U \tag{3.14}
\]
\[
\quad + \alpha\{g(X,Y)\zeta - \theta(Y)X\}
\]
\[
\quad + \beta\{\bar{g}(JX,Y)\zeta - \theta(Y)FX\},
\]
\[
(\nabla_X u)(Y) = -u(Y)\tau(X) - B(X,FY) - \beta\theta(Y)u(X). \tag{3.15}
\]
4 Recurrent and Lie recurrent hypersurfaces

Definition. The structure tensor field $F$ of $M$ is said to be recurrent [6] if there exists a 1-form $\varpi$ on $M$ such that

$$(\nabla_X F)Y = \varpi(X)FY.$$ 

A lightlike hypersurface $M$ of an indefinite trans-Sasakian manifold $\bar{M}$ is called recurrent if it admits a recurrent structure tensor field $F$.

Theorem 4.1. Let $M$ be a recurrent lightlike hypersurface of an indefinite trans-Sasakian manifold $\bar{M}$ with a quarter-symmetric metric connection. Then

1. $F$ is parallel with respect to the induced connection $\nabla$ on $M$,
2. $\bar{M}$ is an indefinite cosymplectic manifold, i.e., $\alpha = \beta = 0$,
3. $D$ and $J(\text{tr}(TM))$ are parallel distributions on $M$, and
4. $M$ is locally a product manifold $\mathcal{C}_U \times M^\sharp$, where $\mathcal{C}_U$ is a null curve tangent to $J(\text{tr}(TM))$ and $M^\sharp$ is a leaf of the distribution $D$.

Proof. (1) As $M$ is recurrent, from the above definition and (3.14), we get

$$\varpi(X)FY = u(Y)A_NX - B(X,Y)U$$

$$+ \alpha\{g(X,Y)\zeta - \theta(X)X\} + \beta\{\bar{g}(JX,Y)\zeta - \theta(Y)FX\}. \tag{4.1}$$

Replacing $Y$ by $\xi$ and using (3.4) and the fact that $F\xi = -V$, we get

$$-\varpi(X)V = \beta u(X)\zeta.$$

Taking the scalar product with $U$ and $\zeta$ by turns, we obtain $\varpi = 0$ and $\beta = 0$ respectively. As $\varpi = 0$, $F$ is parallel with respect to the connection $\nabla$.

(2) Taking $Y = \zeta$ to (4.1) and using (3.9) and $g(FX, \zeta) = 0$, we get

$$\alpha\{-X + u(X)U + \theta(X)\zeta\} = 0.$$ 

It follows that $\alpha = 0$. Thus $\bar{M}$ is an indefinite cosymplectic manifold.

(3) Replacing $Y$ by $U$ to (4.1) and using $\varpi = \alpha = \beta = 0$, we have

$$A_NX = \rho(X)U. \tag{4.2}$$

Taking the scalar product with $V$ to (4.1), we obtain

$$g(A_\xi^*X, Y) = B(X, Y) = u(Y)g(A_NX, V) = g(\rho(X)V, Y).$$

As $A_\xi^*X$ and $V$ belong to $S(TM)$ and $S(TM)$ is non-degenerate, we get

$$A_\xi^*X = \rho(X)V. \tag{4.3}$$
In general, by using (2.1), (2.9), (3.1) and (3.13), we derive
\[ g(\nabla X \xi, V) = -B(X, V), \quad g(\nabla X V, V) = 0, \quad g(\nabla X Z, V) = B(X, FZ), \]
for all \( X \in \Gamma(TM) \) and \( Z \in \Gamma(D_o) \). Taking the scaler product with \( V \) and \( Z \in \Gamma(D_o) \) to (4.3) by turns, we have \( B(X, V) = 0 \) and \( B(X, Z) = 0 \) for all \( X \in \Gamma(TM) \) respectively. As \( FZ \in \Gamma(D_o) \) for \( Z \in \Gamma(D_o) \), it follow that
\[ \nabla X Y \in \Gamma(D), \quad \forall X \in \Gamma(TM), \quad \forall Y \in \Gamma(D). \]
Thus \( D \) is a parallel distribution on \( M \).
Applying \( F \) to (4.2) and using \( FU = 0 \), we get \( F(A_n X) = 0 \). Therefore,
\[ \nabla X U = \tau(X) U, \quad i.e., \]
\[ \nabla X U \in \Gamma(J(\text{tr}(TM))), \quad \forall X \in \Gamma(TM), \]
due to (3.12). Thus \( J(\text{tr}(TM)) \) is also a parallel distribution on \( M \).

(4) As \( D \) and \( J(\text{tr}(TM)) \) are parallel distributions and \( TM = D \oplus J(\text{tr}(TM)) \).
By the decomposition theorem [3], \( M \) is locally a product manifold \( C_u \times M^2 \),
where \( C_u \) is a null curve tangent to \( J(\text{tr}(TM)) \) and \( M^2 \) is a leaf of \( D \).

**Definition.** The structure tensor field \( F \) of \( M \) is said to be \textit{Lie recurrent} [6]
if there exists a 1-form \( \vartheta \) on \( M \) such that
\[ (\mathcal{L}_X F)Y = \vartheta(X)FY, \]
where \( \mathcal{L}_X \) denotes the Lie derivative on \( M \) with respect to \( X \), that is,
\[ (\mathcal{L}_X F)Y = [X, FY] - F[X, Y]. \]
The structure tensor field \( F \) is called \textit{Lie parallel} if \( \mathcal{L}_X F = 0 \). A lightlike hypersurface \( M \) of an indefinite trans-Sasakian manifold \( \bar{M} \) is called \textit{Lie recurrent} if it admits a Lie recurrent structure tensor field \( F \).

**Theorem 4.2.** Let \( M \) be a Lie recurrent lightlike hypersurface of an indefinite trans-Sasakian manifold \( \bar{M} \) with a quarter-symmetric metric connection. Then

(1) \( F \) is Lie parallel,
(2) \( \alpha = 0 \), i.e., \( \bar{M} \) is not an indefinite Sasakian manifold,
(3) the 1-forms \( \theta \) and \( \tau \) satisfy \( d\theta = 0 \) and \( \tau = -\beta \theta \) on \( M \),
(4) the shape operator \( A^*_\xi \) satisfies
\[ A^*_\xi U = 0, \quad A^*_\xi V = 0. \]

**Proof.** (1) Using the above definition, (2.10), (3.2) and (3.14), we get
\[ \vartheta(X)FY = -\nabla_{FY} X + F\nabla_Y X + u(Y)A_n X \]
\[ - \{B(X, Y) - \theta(Y)u(X)\}U - \theta(Y)\{X - \theta(X)\zeta\} \]
\[ + \alpha\{g(X, Y)\zeta - \theta(Y)X\} + \beta\{\bar{g}(JX, Y)\zeta - \theta(Y)FX\}. \]
Taking $Y = \xi$ to (4.6) and using (3.4) and $\theta(\xi) = u(\xi) = 0$, we have
\begin{equation}
-\vartheta(X)V = \nabla_Y X + F\nabla_\xi X + \beta u(X)\zeta. \tag{4.7}
\end{equation}

Taking the scalar product with $V$ and $\zeta$ to (4.7) by turns, we have
\begin{equation}
u(\nabla_Y X) = 0, \quad \theta(\nabla_Y X) = -\beta u(X). \tag{4.8}
\end{equation}

Replacing $Y$ by $V$ to (4.6) and using the fact that $\theta(V) = 0$, we have
\begin{equation}
\vartheta(X)\xi = -\nabla_\xi X + F\nabla_Y X - B(X, V)U + \alpha u(X)\zeta. \tag{4.9}
\end{equation}

Applying $F$ to this equation and using (2.10) and (4.8), we obtain
\begin{equation}
\vartheta(X)V = \nabla_Y X + F\nabla_\xi X + \beta u(X)\zeta.
\end{equation}

Comparing this equation with (4.7), we get $\vartheta = 0$. Thus $F$ is Lie parallel.

(2) Taking the scalar product with $\zeta$ to (4.9), we have
\begin{equation}
g(\nabla_\xi X, \zeta) = \alpha u(X).
\end{equation}

Replacing $X$ by $U$ to this equation and using (3.12), we obtain $\alpha = 0$.

(3) From (1.1), (2.3) and the fact that $\theta(\bar{X}) = \bar{g}(\bar{X}, \zeta)$, we obtain
\begin{equation}
d\theta(\bar{X}, \bar{Y}) = \alpha \bar{g}(\bar{X}, J\bar{Y}). \tag{4.10}
\end{equation}

As $\alpha = 0$, we see that $d\theta = 0$.

Taking the scalar product with $N$ to (4.6) and using (3.6), we have
\begin{equation}
-\bar{g}(\nabla_X Y, N) + \bar{g}(F\nabla_Y X, N) - \theta(Y)\eta(X) - \beta \theta(Y)v(X) = 0. \tag{4.11}
\end{equation}

Replacing $X$ by $\xi$ to (4.11) and using (2.7), (2.8), (2.9) and (3.5), we have
\begin{equation}
B(X, U) + \theta(X) = \tau(FX). \tag{4.12}
\end{equation}

Replacing $X$ by $U$ to (4.12) and using (3.11) and $FU = 0$, we get
\begin{equation}
C(U, V) = B(U, U) = 0. \tag{4.13}
\end{equation}

Replacing $X$ by $V$ to (4.11) and using (2.10), (3.13) and (3.13), we have
\begin{equation}
B(FY, U) = -\tau(Y) - \beta \theta(Y).
\end{equation}

Taking $Y = U$ and $Y = \zeta$ and using the fact that $FU = F\zeta = 0$, we obtain
\begin{equation}
\tau(U) = 0, \quad \tau(\zeta) = -\beta. \tag{4.14}
\end{equation}
Replacing \( X \) by \( U \) to (4.6) and using (2.10), (3.3), and (3.10), we get
\[
u(Y)A_{\eta}\nu U - F(A_{\eta}FY) - A_{\eta}Y - \tau(FY)U - \theta(Y)U + \beta\eta(X)\zeta = 0.
\]
Taking the scalar product with \( V \) and using (3.6), (3.11) and (4.13), we get
\[
B(X,U) + \theta(X) = -\tau(FX).
\]
Comparing this with (4.12), we obtain \( \tau(FX) = 0. \) Replacing \( X \) by \( FY \) to this result and using (2.10) and (4.14), we have \( \tau = -\beta\theta \) on \( M. \)

(4) As \( \tau(X) = \beta\theta(X) \), we have \( \tau(V) = \tau(\xi) = 0. \) Taking \( X = \xi \) to (4.7) and using (3.8), we obtain \( A_{\xi}^{*}V = 0. \) From (3.3) and (4.12), we have \( B(U,X) = 0. \) Thus we obtain \( A_{\xi}^{*}U = 0. \)

5 Indefinite generalized Sasakian space forms

Definition. An indefinite trans-Sasakian manifold \((\bar{M}, J, \zeta, \theta, \bar{g})\) is called an indefinite generalized Sasakian space form, denote it by \( \bar{M}(f_{1}, f_{2}, f_{3}) \), if there exist three smooth functions \( f_{1}, f_{2} \) and \( f_{3} \) on \( \bar{M} \) such that
\[
\bar{R}(X,Y)Z = f_{1}\{\bar{g}(Y,Z)X - \bar{g}(X,Z)Y\}
+ f_{2}\{\bar{g}(X,JZ)JY - \bar{g}(Y,JZ)JX + 2\bar{g}(X,JY)JZ\}
+ f_{3}\{\theta(X)\theta(Z)Y - \theta(Y)\theta(Z)X
+ \bar{g}(X,Z)\theta(Y)\zeta - \bar{g}(Y,Z)\theta(X)\zeta\}.
\]

The generalized Sasakian space form \( \bar{M}(f_{1}, f_{2}, f_{3}) \) was introduced by Alegre et al. [1]. Sasakian, Kenmotsu and cosymplectic space forms are important kinds of indefinite generalized Sasakian space forms such that
\[
f_{1} = \frac{c+3}{4}; \quad f_{2} = f_{3} = \frac{c-1}{4}; \quad f_{1} = \frac{c-3}{4}, f_{2} = f_{3} = \frac{c+1}{4}; \quad f_{1} = f_{2} = f_{3} = \xi,
\]
respectively, where \( c \) is a constant \( J \)-sectional curvature of each space forms.

In this section, we quote the following theorem proved by D.H. Jin [6]:

**Theorem 5.1.** Let \( M \) be a lightlike hypersurface of an indefinite generalized Sasakian space form \( \bar{M}(f_{1}, f_{2}, f_{3}) \) with a quarter-symmetric metric connection. Then the following properties are satisfied

1. \( \alpha \) is a constant,
2. \( \alpha\beta = 0, \)
3. \( f_{1} - f_{2} = \alpha^{2} - \beta^{2} \quad \text{and} \quad f_{1} - f_{3} = (\alpha^{2} - \beta^{2}) + \alpha - \zeta\beta. \)
Comparing the tangential and transversal components of (2.11) and (5.1) and using (3.5) and the fact that $\zeta$ is tangent to $M$, we get

\[
R(X,Y)Z = f_1 \{g(Y,Z)X - g(X,Z)Y\} + f_2 \{\bar{g}(X,JZ)FY - \bar{g}(Y,JZ)FX + 2\bar{g}(X,JY)FZ\} + f_3 \{\theta(X)\theta(Z)Y - \theta(Y)\theta(Z)X + \bar{g}(X,Z)\theta(Y)\zeta - \bar{g}(Y,Z)\theta(X)\zeta\} + B(Y,Z)A_N X - B(X,Z)A_N Y,
\]

\[
(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + \tau(X)B(Y,Z) - \tau(Y)B(X,Z) - \theta(X)B(FY,Z) + \theta(Y)B(FX,Z) = f_2\{u(Y)\bar{g}(X,JZ) - u(X)\bar{g}(Y,JZ) + 2u(Z)\bar{g}(X,JY)\}.
\]

Taking the scalar product with $N$ to (2.11) and using (3.6), we have

\[
\bar{g}(\bar{R}(X,Y)PZ,N) = \bar{g}(R(X,Y)PZ,N).
\]

Substituting (2.13) and (5.1) into the last equation and using (3.2), we get

\[
(\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ) - \tau(X)C(Y,PZ) + \tau(Y)C(X,PZ) - \theta(X)C(FY,PZ) + \theta(Y)C(FX,PZ) = f_1\{g(Y,PZ)\eta(X) - g(X,PZ)\eta(Y)\} + f_2\{v(Y)\bar{g}(X,JPZ) - v(X)\bar{g}(Y,JPZ) + 2v(PZ)\bar{g}(X,JY)\} + f_3\{\theta(X)\eta(Y) - \theta(Y)\eta(X)\}\theta(PZ).
\]

**Theorem 5.2.** Let $M$ be a recurrent lightlike hypersurface of an indefinite generalized Sasakian space form $\tilde{M}(f_1, f_2, f_3)$ with a quarter-symmetric metric connection. Then $\tilde{M}(f_1, f_2, f_3)$ is flat, i.e., $f_1 = f_2 = f_3 = 0$, $M$ is also flat, and the 1-form $\tau$ is closed on $M$, i.e., $d\tau = 0$.

**Proof.** As $M$ is recurrent, by Theorem 4.1, we show that $\alpha = \beta = 0$. Thus, by (3) of Theorem 5.1, we see that $f_1 = f_2 = f_3 = 0$.

Taking the scalar product with $U$ to (4.2), we have

\[
C(X,U) = 0.
\]

Applying $\nabla_Y$ to this equation and using (4.4), we get

\[
(\nabla_X C)(Y,U) = 0.
\]

Replacing $PZ$ by $U$ to (5.4) and using the last two equations, we obtain

\[
(f_1 + f_2)\{v(Y)\eta(X) - v(X)\eta(Y)\} = 0.
\]
Taking $X = \xi$ and $Y = V$, we have $f_1 + f_2 = 0$. Therefore $f_1 = f_2 = f_3 = 0$ and $\bar{M}(f_1, f_2, f_3)$ is flat. Taking the scalar product with $Y$ to (4.3), we get

$$B(X, Y) = \rho(X)u(Y).$$

(5.5)

Substituting (4.2) and (5.5) into (5.2) with $f_1 = f_2 = f_3 = 0$, we have

$$R(X, Y)Z = \{\rho(Y)\rho(X) - \rho(X)\rho(Y)\}u(Z)U = 0.$$

Thus $M$ is also flat. From (4.4), we get

$$R(X, Y)U = 2d\tau(X, Y)U.$$ Thus we see that $d\tau = 0$.

**Theorem 5.3.** Let $M$ be a Lie recurrent lightlike hypersurface of an indefinite generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ with a quarter-symmetric metric connection. Then $\bar{M}(f_1, f_2, f_3)$ is a space form such that

$$f_1 = -\beta^2, \quad f_2 = 0, \quad f_3 = \zeta \beta.$$

*Proof.* Replacing $Z$ by $\xi$ to (5.2) and then, comparing with (2.14), we have

$$\nabla_X^*(A^*_\xi Y) + \nabla_Y^*(A^*_\xi X) + A^*_\xi [X, Y] - \tau(X)A^*_\xi Y + \tau(Y)A^*_\xi X$$

$$+ \{C(Y, A^*_\xi X) - C(X, A^*_\xi Y) - 2d\tau(X, Y)\}\xi$$

$$= f_2\{u(Y)FX - u(X)FY - 2\bar{g}(X, JY)V\}.$$

Taking the scalar product with $N$, we obtain

$$C(Y, A^*_\xi X) - C(X, A^*_\xi Y) - 2d\tau(X, Y) = f_2\{v(X)u(Y) - v(Y)u(X)\}.$$

Taking $X = V$ and $Y = U$ to this equation and using (4.5), we obtain

$$-2d\tau(V, U) = f_2.$$

From the facts that $\tau = \beta \theta$ and $d\theta = 0$, we obtain

$$2d\tau(X, Y) = (Y \beta)\theta(X) - (X \beta)\theta(Y).$$

Thus $d\tau(V, U) = 0$ and $f_2 = 0$. By Theorem 5.1, we get $f_1 = -\beta^2$ and $f_3 = \zeta \beta$.

**References**

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