Geometric Construction of an Ellipse
from Its Moments

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Abstract

This paper offers a trio of Euclidean constructions of an ellipse given its first moments and second central moments. The crux of these constructions lies in the graphical construction of Carlyle [4, p. 99] and Lill [2] for the roots of a quadratic and the Pappus-Euler problem [12] of the construction of the principal axes of an ellipse from a pair of its conjugate diameters.

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1 Preliminaries

An ellipse \( \mathcal{E} \) is uniquely determined by its first moments

\[
M_x := \frac{1}{\mathcal{A}_\mathcal{E}} \int \int_{\mathcal{E}} x \, dx \, dy, \quad M_y := \frac{1}{\mathcal{A}_\mathcal{E}} \int \int_{\mathcal{E}} y \, dx \, dy,
\]

(1)

together with its second central moments

\[
M_{xx} := \frac{1}{\mathcal{A}_\mathcal{E}} \int \int_{\mathcal{E}} (x - M_x)^2 \, dx \, dy,
\]

(2)

\[
M_{xy} := \frac{1}{\mathcal{A}_\mathcal{E}} \int \int_{\mathcal{E}} (x - M_x)(y - M_y) \, dx \, dy,
\]

(3)
\[ M_{yy} := \frac{1}{A_E} \int \int_{E} (y - M_y)^2 \, dx \, dy, \]  

where \( A_E \) is the area of the ellipse. Of course, the inequalities \( M_{xx} > 0, M_{yy} > 0 \) and \( M_{xy}^2 < M_{xx}M_{yy} \) must be satisfied.

Specifically [8], the equation of the required ellipse is given by the positive definite quadratic form

\[
\begin{bmatrix} x - M_x & y - M_y \end{bmatrix} \begin{bmatrix} M_{xx} & M_{xy} \\ M_{xy} & M_{yy} \end{bmatrix}^{-1} \begin{bmatrix} x - M_x \\ y - M_y \end{bmatrix} = 4, \]

and the elliptical area may be expressed as

\[ A_E = 4\pi \sqrt{M_{xx}M_{yy} - M_{xy}^2}. \]

The slopes of the principal axes of the ellipse are the roots of the quadratic

\[ m^2 + \frac{M_{xx} - M_{yy}}{M_{xy}} \cdot m - 1 = 0, \]

so that

\[
m_{\text{major}} = \frac{(M_{yy} - M_{xx}) + \sqrt{(M_{yy} - M_{xx})^2 + 4M_{xy}^2}}{2M_{xy}},
\]

\[
m_{\text{minor}} = \frac{(M_{yy} - M_{xx}) - \sqrt{(M_{yy} - M_{xx})^2 + 4M_{xy}^2}}{2M_{xy}},
\]

since \( m_{\text{major}} \) has the same sign as \( M_{xy} \) [6].

The corresponding squared magnitudes of the principal semiaxes [15, p. 158] are the roots of the quadratic

\[ \sigma^2 - 4(M_{xx} + M_{yy}) \cdot \sigma + 16(M_{xx}M_{yy} - M_{xy}^2) = 0, \]

so that

\[
\sigma_{\text{major}}^2 = 2 \left[ (M_{xx} + M_{yy}) + \sqrt{(M_{xx} - M_{yy})^2 + 4M_{xy}^2} \right],
\]

\[
\sigma_{\text{minor}}^2 = 2 \left[ (M_{xx} + M_{yy}) - \sqrt{(M_{xx} - M_{yy})^2 + 4M_{xy}^2} \right].
\]

Equation (5) may be expanded thereby yielding

\[ M_{yy}(x - M_x)^2 - 2M_{xy}(x - M_x)(y - M_y) + M_{xx}(y - M_y)^2 = 4(M_{xx}M_{yy} - M_{xy}^2) \]

as the equation of the ellipse. Setting \( y = M_y \) yields the corresponding horizontal points of intersection

\[ (M_x \pm 2\sqrt{M_{xx}M_{yy} - M_{xy}^2}, M_y), \]
while setting $x = M_x$ yields the corresponding vertical points of intersection

$$\left( M_x, M_y \pm 2\sqrt{M_{yy} - M_{xy}^2/M_{xx}} \right).$$

The points of vertical and horizontal tangency are [7], respectively,

$$\left( M_x \pm 2\sqrt{M_{xx}}, M_y \pm 2\sqrt{M_{yy}} \right); \quad \left( M_x \pm 2\sqrt{M_{xx}}, M_y \pm 2\sqrt{M_{yy}} \right).$$

![Figure 1: Conjugate Semidiameters](image)

With reference to Figure 1, recall that two semidiameters are conjugate to one another if they are parallel to each others tangents [8]. For this to occur, it is necessary and sufficient that their slopes $m$ and $s$ satisfy the symmetric conjugacy condition: [7]

$$M_{xx} \cdot (m \cdot s) - M_{xy} \cdot (m + s) + M_{yy} = 0.$$

Thus, the semidiameter to a point of vertical/horizontal tangency is conjugate to the semidiameter to a point of vertical/horizontal intersection, respectively. (See Figures 4/5, Left.)

In the succeeding sections, a trio of straightedge-compass constructions of an ellipse, $\mathcal{E}$, given its moments, $(M_x, M_y, M_{xx}, M_{xy}, M_{yy})$, will be developed. Strictly speaking, such a Euclidean construction is not possible for the ellipse.
However, given the principal axes of the ellipse, several methods are available for drawing any number of points on the ellipse [3]. Thus, the phrase “construct the ellipse” will be interpreted as “construct the principal axes of the ellipse”.

Certain primitive arithmetic/algebraic Euclidean constructions [1, pp. 121-122], as next described, will be required.

**Proposition 1 (Primitive Constructions)** Given two line segments of non-zero lengths \(a\) and \(b\), one may construct line segments of length \(a + b\), \(a - b\), \(a \cdot b\), \(a/b\) and \(\sqrt{a}\) using only straightedge and compass.

2 The Carlyle-Lill Construction

![Figure 2: Determination of Axes Directions](image)

Because Equations (8-9) involve only the operators \(+, -, \cdot, /, \sqrt{}\), application of Proposition 1 would provide an immediate construction of the directions of the principal axes. Furthermore, since Equations (11-12) also involve only this same set of operators, an additional application would yield the squared magnitudes of the principal semi-axes thereby leading to their complete construction. Clearly, an approach less reminiscent of a sledgehammer is to be preferred.

Instead, the following beautiful construction due to Carlyle [4, p. 99] and Lill [2, pp. 355-356] will be invoked.

**Proposition 2 (Graphical Solution of a Quadratic Equation)** Draw a circle having as diameter the line segment joining \((0, 1)\) and \((a, b)\). The abscissae of the points of intersection of this circle with the \(x\)-axis are the roots of the quadratic \(x^2 - ax + b = 0\).
Construction of an ellipse from moments

A double application of Proposition 2, in concert with Equations (7) and (10), produces the following far more elegant construction.

Construction 1 (Double Carlyle-Lill Construction)  Given the first and second moments \( \langle M_x, M_y, M_{xx}, M_{xy}, M_{yy} \rangle \):

1. According to Equation (7), apply the Carlyle-Lill construction, Proposition 2, with \((a, b) = (M_{yy} - M_{xx}, -1)\) to construct the directions of the principal axes. (See Figure 2.)

2. According to Equation (10), apply the Carlyle-Lill construction, Proposition 2, with \((a, b) = (4(M_{xx} + M_{yy}), 16(M_{xx}M_{yy} - M_{xy}^2))\) to construct the squared magnitudes of the principal semiaxes. (See Figure 3, Left.)

3. Apply the \( \sqrt{\cdot} \)-operator, Proposition 1, to these squared magnitudes in order to construct the magnitudes, \( \{\sigma_{\text{major}}, \sigma_{\text{minor}}\} \), of the principal semiaxes emanating from the centroid, \((M_x, M_y)\). (See Figure 3, Right.)

4. From the principal semiaxes, \( \{\bar{\sigma}_{\text{major}}, \bar{\sigma}_{\text{minor}}\} \), a variety of methods [3] may now be employed to construct any number of points lying on the ellipse

\[
\mathcal{E} : \vec{r}(t) = \cos \theta \cdot \bar{\sigma}_{\text{major}} + \sin \theta \cdot \bar{\sigma}_{\text{minor}} \quad (0 \leq \theta < 2\pi).
\]
The following problem was first formulated and solved by Pappus in the 4th Century, although Euler gave the first proof of its validity only in the 18th Century [9].

**Proposition 3 (Pappus-Euler Problem)** Given two conjugate diameters of an ellipse, to find the principal axes in both position and magnitude.

As Euler’s demonstration of Pappus’ construction was purely synthetic in nature, McCartin subsequently provided both a purely vector analytic demonstration [10] as well as an independent matrix-vector analytic demonstration [11].

Since that time, many alternative solutions have been devised [5, 14]. Most recently, McCartin has provided a matrix-vector analytic construction [12] which will next be utilized, in tandem with some prior observations, to provide a pair of related constructions of an ellipse from its given moments.

As previously noted, the semidiameter of a point of vertical intersection is conjugate to the semidiameter to a point of vertical tangency. Thus, by Equations (15) and (16), the vectors

\[ \langle 0, 2\sqrt{M_{yy} - M_{xy}^2/M_{xx}} \rangle; \langle 2\sqrt{M_{xx}}, 2\frac{M_{xy}}{\sqrt{M_{xx}}} \rangle \]

(18)

define a pair of conjugate semidiameters emanating from the centroid, \((M_x, M_y)\), which may be utilized in the Pappus-Euler problem, Proposition 3, to construct the principal axes of the ellipse. (See Figure 4.)
Construction 2 (Vertical Tangent Construction) Given the first and second moments $\langle M_x, M_y, M_{xx}, M_{xy}, M_{yy} \rangle$, apply the McCartin algorithm [12] to the pair of conjugate semidiameters

$$\langle 0, 2\sqrt{M_{yy} - M_{xy}^2/M_{xx}} \rangle; \langle 2\sqrt{M_{xx}}M_{xy}/\sqrt{M_{xx}}, M_{xy} \rangle$$

thereby constructing the principal axes of the ellipse in direction and magnitude emanating from the centroid, $(M_x, M_y)$.

Likewise, the semidiameter of a point of horizontal intersection is conjugate to the semidiameter to a point of horizontal tangency. Thus, by Equations (14) and (16), the vectors

$$\langle 2\sqrt{M_{yy} - M_{xy}^2/M_{xx}}, 0 \rangle; \langle 2\sqrt{M_{xx}}, 2\sqrt{M_{yy}} \rangle$$

(19)

define a pair of conjugate semidiameters emanating from the centroid, $(M_x, M_y)$, which may be utilized in the Pappus-Euler problem, Proposition 3, to construct the principal axes of the ellipse. (See Figure 5.)

Construction 3 (Horizontal Tangent Construction) Given the first and second moments $\langle M_x, M_y, M_{xx}, M_{xy}, M_{yy} \rangle$, apply the McCartin algorithm [12] to the pair of conjugate semidiameters

$$\langle 2\sqrt{M_{yy} - M_{xy}^2/M_{xx}}, 0 \rangle; \langle 2\sqrt{M_{xx}}M_{xy}/\sqrt{M_{yy}}, 2\sqrt{M_{yy}} \rangle$$
thereby constructing the principal axes of the ellipse in direction and magnitude emanating from the centroid, \((M_x, M_y)\).

4 Postscript

In the foregoing, a trio of Euclidean constructions for an ellipse given its first moments and second central moments was developed. The first such construction relied heavily upon the Carlyle-Lill graphical construction of the roots of a quadratic equation. The remaining constructions hinged upon the many available solutions to the Pappus-Euler problem for the construction of the principal axes of an ellipse given any pair of its conjugate diameters [5, 9, 10, 11, 12, 14].

While the present paper was focused on continuous distributions of mass, given the appropriate first and second moments, the very same constructions are applicable to the construction of the “concentration ellipse” [6] associated with a discrete distribution of points. Furthermore, this observation permits the unified construction of various lines of regression [13].

References


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