On the Approximate Evaluation of
Real Singular and Strongly Singular Integrals

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Abstract

Some derivative free quadrature rules for the numerical evaluation of real singular and strongly singular integrals have been proposed in this paper. Rules have been tested numerically by some standard test integrals and their asymptotic error bounds have been determined.

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Keywords: singularity; Cauchy principal value; residue; asymptotic error bound

1 Introduction

The strongly singular integral of the type

\[ I(f) = \int_{-a}^{a} \frac{f(x)}{(x - \xi)^2} dx; -a < \xi < a; \]  

where \( f(x) \) is a well behaved function appears frequently in many branches of physics, aerodynamics, the singular eigen function method in neutron transport, crack problem in plane elasticity and scattering theory etc. For instance in the formulation of simple crack problem (Ref.[1]) on the whole \( xy \)-plane
containing a finite cut (crack) in the region $-a < x < a; y = 0$; the solution satisfies the two dimensional Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0; \quad (2)$$

subject to condition

$$\frac{\partial u}{\partial y}|_{y=0} = g(x); -a < x < a;$$

$$u \to 0 \ as \ (x^2 + y^2) \to \infty; \quad (3)$$

where $g(x)$ is a suitably given function. It is seen that (Ref.[2]) when condition (3) is incorporated during the solution of the above physical problem, it leads to a divergent integral of the type (1). Researchers like Ramm and Van der Sluis[3], Groetsch[4], Criscuolo[5] and many more as available in literature have been contributed their work for the approximate evaluation of this type of integrals.

Further the Cauchy principal value integral

$$J(f) = P \int_{-a}^{a} \frac{f(x)}{(x - \xi)} \, dx; -a < \xi < a; \quad (4)$$

appears in the formulation of many physical sciences and engineering problems. Some of the illustrious researchers: Diethelm[6], Bej and et al[8], Das and Hota[7] have formulated quadrature rules for its numerical approximations and also many are still engaged in the study of this type of integrals in order to frame quadrature rules in general and specifically in case of specific integrands. The mathematical relationship between equations (1) and (4) is observed as

$$\frac{d}{d\xi}[P \int_{-\xi}^{\xi} \frac{f(x)}{(x - \xi)} \, dx] = I(f). \quad (5)$$

However it is seen that the quadrature rules meant for the numerical integration of the integral $J(f)$ behave very much unstable when these are applied for the approximate evaluation of the integral $I(f)$ due to the presence higher order singularity in the interval of integration. So, separate type of treatments have been adopted by researchers with different class of quadrature rules separately for these two type of singular integrals as given in equations (1) and (4); though these two have a strong mathematical relationship. Since the integral $J(f)$ with the transformation $t = x - \xi$ is transformed to

$$J(f) = \int_{-a-\xi}^{a-\xi} \frac{f(t + \xi)}{t} \, dt; |\xi| < a; \xi \neq 0;$$
can be rewritten as sum of the non-singular integral

\[ J_1(f) = \int_{-a}^{a} \frac{f(t + \xi)}{t} dt; \]

(where standard quadrature rules such as Newton-Cotes type or Gauss-Legendre type of rules can be applied for its numerical approximations) and the singular integral

\[ J_2(f) = \int_{-a}^{a} \frac{f(t + \xi)}{t} dt; \]

having singularity at origin; thus in this paper we have developed some quadrature rules of increasing algebraic degree of precision for the numerical integration of the CPV integral of the type

\[ T(f) = \int_{-a}^{a} \frac{f(x)}{x} dx; \]

what in latter stages can be applied for the numerical integrations of the integrals of the type \( J(f) \) and \( I(f) \) as given in equations (4) and (1) respectively. Also we have applied the proposed quadrature rules for the approximations of the non-singular integral \( J_1(f) \) in order to measure the efficiency of our rules numerically.

2 Formulation of Rules

The proposed \((4n - 1)\)-point rule denoted by \( R_n(f) \) based on the nodes

\[ 0; \pm \frac{a}{2n}; \pm \frac{2a}{2n}; \pm \frac{3a}{2n}; \ldots; \pm \frac{(2n-1)a}{2n}; \]

is defined as

\[ R_n(f) = w_{n0}f(0) + \sum_{k=1}^{(2n-1)} w_{nk} \left[ f \left( \frac{ka}{2n} \right) - f \left( -\frac{ka}{2n} \right) \right]. \]  

(6)

Since the nodes are prefixed, it only remains to determine the coefficients \( w_{n0} \) and \( w_{nk} \); for \( k = 1(1)(2n - 1) \) associated with \( f(0) \) and with the block

\[ \left[ f \left( \frac{ka}{2n} \right) - f \left( -\frac{ka}{2n} \right) \right]; \]

respectively. It is to be noted here that in such rules the coefficient of \( f(0) \) i.e. \( w_{n0} \) is zero, for all \( n \). The method of undetermined coefficients has been adopted to determine the coefficients \( w_{nk} \) of the above rule \( R_n(f) \) for all \( n \) based on the following definition.
Degree of Precision: We say that \( d \geq 0 \) is the degree of precision of the rule \( R_n(f) \) if
\[
1. P \int_{-a}^{a} \frac{x^k}{x} dx = R_n(x^k); \forall k = 0(1)d;
\]
\[
2. P \int_{-a}^{a} \frac{x^{d+1}}{x} dx \neq R_n(x^{d+1});
\]
holds. Now based on the above definition the coefficients \( w_{nk} \); for \( k = 0(1)(2n-1) \) of the rules \( R_n(f) \) for all \( n \), in general are the solutions of the following set of moment equations
\[
AW = B;
\]
where
\[
A = \begin{pmatrix}
1 & 2 & \ldots & (2n-1)
\\
1^3 & 2^3 & \ldots & (2n-1)^3
\\
1^5 & 2^5 & \ldots & (2n-1)^5
\\
\vdots & \vdots & \ldots & \vdots
\\
1^{2n-1} & 2^{2n-1} & \ldots & (2n-1)^{2n-1}
\end{pmatrix};
\]
\[
W = \begin{pmatrix}
w_{n1}
\\
w_{n2}
\\
\vdots
\\
w_{n(2n-1)}
\end{pmatrix};
\]
\[
B = \begin{pmatrix}
2n/1 \\
(2n)^3/3
\\
\vdots
\\
(2n)^{2n-1}
\end{pmatrix}.
\]
The weights and the rules corresponding to \( n = 1, 2, 3 \) and 4 are noted below.
\[
R_1(f) = 2 \left[ f \left( \frac{a}{2} \right) - f \left( -\frac{a}{2} \right) \right];
\]
\[
R_2(f) = \frac{134}{45} \left[ f \left( \frac{a}{4} \right) - f \left( -\frac{a}{4} \right) \right] - \frac{206}{225} \left[ f \left( \frac{a}{2} \right) - f \left( -\frac{a}{2} \right) \right] + \frac{214}{225} \left[ f \left( \frac{3a}{4} \right) - f \left( -\frac{3a}{4} \right) \right];
\]
\[
R_3(f) = \frac{4433}{525} \left[ f \left( \frac{a}{6} \right) - f \left( -\frac{a}{6} \right) \right] - \frac{2717}{337} \left[ f \left( \frac{a}{3} \right) - f \left( -\frac{a}{3} \right) \right] + \frac{14779}{2450} \left[ f \left( \frac{a}{2} \right) - f \left( -\frac{a}{2} \right) \right] - \frac{422}{201} \left[ f \left( \frac{2a}{3} \right) - f \left( -\frac{2a}{3} \right) \right] + \frac{489}{614} \left[ f \left( \frac{5a}{6} \right) - f \left( -\frac{5a}{6} \right) \right];
\]
and
\[
R_4(f) = \frac{14687}{489} \left[ f \left( \frac{a}{8} \right) - f \left( -\frac{a}{8} \right) \right] - \frac{4316}{63} \left[ f \left( \frac{a}{4} \right) - f \left( -\frac{a}{4} \right) \right] + \frac{4044}{71} \left[ f \left( \frac{3a}{8} \right) - f \left( -\frac{3a}{8} \right) \right] - \frac{4376}{139} \left[ f \left( \frac{a}{2} \right) - f \left( -\frac{a}{2} \right) \right] + \frac{2927}{232} \left[ f \left( \frac{5a}{8} \right) - f \left( -\frac{5a}{8} \right) \right] - \frac{1124}{357} \left[ f \left( \frac{3a}{4} \right) - f \left( -\frac{3a}{4} \right) \right] + \frac{485}{671} \left[ f \left( \frac{7a}{8} \right) - f \left( -\frac{7a}{8} \right) \right].
\]
In the light of the definition of the degree of precision of the rules as given above are of precision 2, 6, 10 and 14 respectively. In general the degree of precision of the formulated open type rule \( R_n(f) \) is \( 4n - 2 \).
2.1 Scheme for the Numerical Approximation of strongly singular Integrals

The strongly singular integral \( I(f)(\text{equation}(1)) \) can also be transformed into the sum of a non-singular integral and a singular integral with higher order singularity at origin by the same transformation as suggested above. Since the class of rules that have been constructed for numerical integration of the Cauchy principal value of integrals given in equations (7) to (10) can’t be readily applied for the approximation of singular integral of type

\[
\Gamma(f) = H \int_{-a}^{a} \frac{f(x)}{x^2} \, dx; \tag{11}
\]

therefore, here we propose a scheme for numerical evaluation of the integral as given above by reducing the order of its singularity from two to one; and following the scheme, we present some integrals of the type \( I(f) \) have been successfully numerically integrated.

**Scheme of Evaluation:**

For the construction of the scheme, here we assume that \( f(z) \) is the analytic continuation of \( f(x) \) in the disc \( \Omega = \{ z \in C : |z| \leq r ; r > a \} \). As a result,

\[
f(z) = f(x); \forall x \in [-a,a].
\]

Since \( f(x) \) is well behaved in \([-a,a]\), thus by Taylor’s theorem we obtain

\[
\frac{f(x) - c_0}{x} = c_1 + c_2x + c_3x^2 + c_4x^3 + \ldots
\]

Further since

\[
\frac{f(x) - c_0}{x^2} = \frac{f(x)-c_0}{x} = \frac{g(x)}{x};
\]

where

\[
g(x) = c_1 + c_2x + c_3x^2 + c_4x^3 + \ldots;
\]

is analytic on \( \Omega \), thus the integral

\[
P \int_{-a}^{a} \frac{g(x)}{x} \, dx;
\]

is a Cauchy principal value integral and can be approximated by the rules as constructed in this paper. Therefore integral given in (11) can be transformed into

\[
\Gamma(f) = H \int_{-a}^{a} \frac{f(x)}{x^2} \, dx = K + L;
\]

where

\[
K = \int_{-a}^{a} \frac{(f(x)-c_0)}{x} \, dx; c_0 = \text{Res} \left\{ \frac{f(x)}{x} \right\}
\]

\[
= \int_{-a}^{a} \frac{g(x)}{x} \, dx \simeq R_n(g); g(x) = \left( \frac{f(x) - c_0}{x} \right)
\]
and
\[ L = \int_{-a}^{a} \frac{c_0}{x^2} \, dx = \frac{2c_0}{a}. \]

Now since the integral \( K \) is a singular integral of the type (4), thus the rules \( R_1(f) \) to \( R_4(f) \) as given from equations (7) to (10) meant for the numerical integration of the real CPV integrals may be applied for its numerical approximations. As a result the integral
\[ \Gamma(f) = H \int_{-a}^{a} \frac{f(x)}{x^2} \, dx \simeq R_n(g) - \frac{2c_0}{a}; \]

where \( g(x) = \left( \frac{f(x) - c_0}{x} \right) \); and \( a \) is the one of the end points of the interval of integration \([-a, a]\).

### Asymptotic Error Estimates

The first leading term of the asymptotic error expression associated with all the four rules \( R_1(f), R_2(f), R_3(f) \) and \( R_4(f) \) meant for numerical evaluation of real CPV integrals in order of their increasing accuracy are given in Table 1. From these tabulated values one may be concluded that the rule \( R_4(f) \) is the rule of maximum accuracy among all other rules of its class.

<table>
<thead>
<tr>
<th>Rules</th>
<th>Asymptotic Error expressions</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R_1(f) )</td>
<td>( 0.03 \times f^{(3)}(0) )</td>
</tr>
<tr>
<td>( R_2(f) )</td>
<td>( 9.8 \times 10^{-6} f^{(r)}(0) )</td>
</tr>
<tr>
<td>( R_3(f) )</td>
<td>( 2.6 \times 10^{-10} f^{(11)}(0) )</td>
</tr>
<tr>
<td>( R_4(f) )</td>
<td>( 1.8 \times 10^{-15} f^{(15)}(0) )</td>
</tr>
</tbody>
</table>

Table 1: Asymptotic Error Estimates of the Rules \( R_n(f) \); \( \forall n = 1, 2, 3 \) and 4 for \( a = 1 \).

### 3 Numerical Experiments

In this section we have applied our proposed quadrature scheme for the numerical integrations of the integrals
\[ J(f) = P \int_{-1}^{1} \frac{f(x)}{(x-\xi)} \, dx \quad \text{and} \quad I(f) = H \int_{-1}^{1} \frac{f(x)}{(x-\xi)^2} \, dx; \]

for different values of \(-1 < \xi < 1\). The results of their numerical integrations are given in Table 2.
On approximate evaluation ...

<table>
<thead>
<tr>
<th>Rules</th>
<th>ξ</th>
<th>Approx. of $I=\int_{-1}^{1} \frac{e^{x}}{\xi} dx$</th>
<th>Abs. Err</th>
<th>Approx. of $J=\int_{-1}^{1} \frac{e^{x}}{\xi^{2}} dx$</th>
<th>Abs. Err</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_1$</td>
<td>0</td>
<td>2.08438122197</td>
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<td>-0.978992278349</td>
<td>$7.3 \times 10^{-3}$</td>
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<td>$9.3 \times 10^{-6}$</td>
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<td>$1.2 \times 10^{-6}$</td>
</tr>
<tr>
<td>$R_3$</td>
<td></td>
<td>2.11450175049</td>
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<td>-0.971659518901</td>
<td>$2.2 \times 10^{-11}$</td>
</tr>
<tr>
<td>$R_4$</td>
<td></td>
<td>2.11450175075</td>
<td>0.0</td>
<td>-0.971659518879</td>
<td>0.0</td>
</tr>
<tr>
<td>Exact value</td>
<td></td>
<td>2.11450175075</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$R_1$</td>
<td>$1/4$</td>
<td>1.7259302686</td>
<td>$1.2 \times 10^{-2}$</td>
<td>-2.1925879454</td>
<td>$1.2 \times 10^{-2}$</td>
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<tr>
<td>$R_2$</td>
<td></td>
<td>1.7382469604</td>
<td>$2.4 \times 10^{-6}$</td>
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<tr>
<td>$R_3$</td>
<td></td>
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<tr>
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<td>0.0</td>
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<td>0.0</td>
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<tr>
<td>Exact value</td>
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</tr>
<tr>
<td>$R_1$</td>
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<td>-2.8502151990</td>
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<tr>
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<td>$5.0 \times 10^{-9}$</td>
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<tr>
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<td></td>
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</table>

Table 2: Approx. of real singular and strongly singular integrals

References


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