Certain Identities Associated with 3-Dissection Property, Continued-Fraction and Combinatorial Partition

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Abstract

The objective of this paper is to establish certain (presumably) new identities which depict interrelationships between 3-dissection property, combinatorial partition and continued fraction, by using some easily-derivable identities for $q$-products and combining some earlier results.

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1 Introduction and Preliminaries

Throughout this paper, \( \mathbb{N} \), \( \mathbb{Z} \), and \( \mathbb{C} \) denote the sets of positive integers, integers, and complex numbers, respectively, and \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \). For the sake of easy reference, we recall the following \( q \)-notations (see, e.g., [8, Chapter 6]).

The \( q \)-shifted factorial \((a; q)_n\) is defined by

\[ (a; q)_n := \begin{cases} 1, & (n = 0) \\ \prod_{k=0}^{n-1} (1 - a q^k), & (n \in \mathbb{N}) \end{cases} \tag{1} \]

where \( a, q \in \mathbb{C} \) and it is assumed that \( a \neq q^{-m} \) \((m \in \mathbb{N}_0)\). We also write

\[ (a; q)_\infty := \prod_{k=0}^{\infty} (1 - a q^k) \tag{2} \]

\[ = \prod_{k=1}^{\infty} (1 - a q^{k-1}) \quad (a, q \in \mathbb{C}; \ |q| < 1). \]

It is noted that, when \( a \neq 0 \) and \( |q| \geq 1 \), the infinite product in (2) diverges. So, whenever \((a; q)_\infty\) is involved in a given formula, the constraint \( |q| < 1 \) will be tacitly assumed.

The following notations are also frequently used:

\[ (a_1, a_2, \cdots, a_m; q)_n := (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n \tag{3} \]

and

\[ (a_1, a_2, \cdots, a_m; q)_\infty := (a_1; q)_\infty (a_2; q)_\infty \cdots (a_m; q)_\infty \tag{4} \]

In 2010, Chan [1] investigated Ramanujan’s cubic continued fraction and the coefficients \( a(n) \) defined by the following generating function:

\[ \sum_{n=0}^{\infty} a(n) q^n = \frac{1}{(q; q)_\infty (q^2; q^2)_\infty}. \tag{5} \]

Chan [1] presented the generating function of \( a(3n + 2) \) as follows:

\[ \sum_{n=0}^{\infty} a(3n + 2) q^n = 3 \left\{ (q^3; q^3)_\infty \right\}^3 \left\{ (q^6; q^6)_\infty \right\}^3 \left\{ (q; q)_\infty \right\}^4 \left\{ (q^2; q^2)_\infty \right\}^4, \tag{6} \]

which was also proved by Cao [3] who used the 3-dissection for \((q; q)_\infty (q^2; q^2)_\infty\).

In 2011, Zhao and Zhong [2] investigated arithmetic properties of the functions \( b(n) \) whose generating function is the square of the generating function in (5), that is,

\[ \sum_{n=0}^{\infty} b(n) q^n = \frac{1}{\left\{ (q; q)_\infty \right\}^2 \left\{ (q^2; q^2)_\infty \right\}^2}. \tag{7} \]
3-Dissection property, continued-fraction identities

It is noted that many properties of \( a(n) \) and \( b(n) \) are similar to those of the standard partition function \( p(n) \) which is defined to be the number of ways of writing \( n \) as a sum of positive integers in non-increasing order. It is conventionally assumed that \( p(n) = 0 \) \((n \in \mathbb{Z} \setminus \mathbb{N})\). In this sense, the formula (6) can be considered as an analogue of Ramanujan’s most beautiful identity:

\[
\sum_{n=0}^{\infty} p(5n+4) q^n = 5 \frac{(q^5; q^5)_{\infty}}{(q; q)_{\infty}}.
\]

Ramanujan defined general theta function \( f(a, b) \) as follows (see, for details, [5, p. 31, Eq. (18.1)]):

\[
f(a, b) := 1 + \sum_{n=1}^{\infty} (ab)^{-\frac{n(n+1)}{2}} (a^n + b^n) = \sum_{n=-\infty}^{\infty} a^{-\frac{n(n+1)}{2}} b^{-\frac{n(n-1)}{2}} = f(b, a) \quad (|ab| < 1).
\]

It is easy to see that

\[
f(a, b) = a^{-\frac{n(n+1)}{2}} b^{-\frac{n(n-1)}{2}} f(a(ab)^n, b(ab)^n) = f(b, a) \quad (n \in \mathbb{Z}).
\]

Ramanujan also rediscovered Jacobi’s famous triple-product identity (see [5, p. 35, Entry 19]):

\[
f(a, b) = (-a; ab)_{\infty}(-b; ab)_{\infty}(ab; ab)_{\infty},
\]

which was, in fact, first proved by Gauss.

Several \( q \)-series identities emerging from Jacobi’s triple-product identity (11) are worthy of note here (see [5, pp. 36-37, Entry 22]):

\[
\phi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} = 1 + 2 \sum_{n=1}^{\infty} q^{n^2} = \frac{(-q; q^2)_{\infty}(q^2; q^2)_{\infty}}{(q^2; q^2)_{\infty} (-q^2; q^2)_{\infty}};
\]

\[
\psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{-\frac{n(n+1)}{2}} = \frac{(q^2; q^2)_{\infty}}{(q; q^3)_{\infty}};
\]

\[
f(-q) = f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{-\frac{n(3n-1)}{2}} = \sum_{n=0}^{\infty} (-1)^n q^{-\frac{n(3n+1)}{2}} = (q; q)_{\infty}.
\]
The identity (14) is known as Euler's pentagonal number theorem. The following \( q \)-series identity:
\[
(-q; q)_\infty = \frac{1}{(q; q^2)_\infty}
\]
provides the analytic equivalence of Euler's famous theorem.

It is known that the number of partitions of \( n \in \mathbb{N} \) into distinct parts is equal to the number of partitions of \( n \) into odd parts.

We now recall the Rogers-Ramanujan continued fraction \( R(q) \) given by
\[
R(q) := \frac{q^\frac{1}{4} H(q)}{G(q)} = q^\frac{1}{4} \frac{f(-q, -q^4)}{f(-q^2, -q^3)} = q^\frac{1}{4} \frac{(q; q^5)_\infty (q^4; q^5)_\infty}{(q; q^2)_\infty (q^3; q^5)_\infty}
\]
\[
= \frac{q^\frac{1}{4} q q^2 q^3}{1 + 1 + 1 + 1 + \cdots} \quad (|q| < 1)
\]
in terms of the following widely-investigated Roger-Ramanujan identities:
\[
G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_\infty} = \frac{f(-q^5)}{f(-q^2; -q^3)} = \frac{1}{(q; q^5)_\infty (q^4; q^5)_\infty (q^5; q^5)_\infty} \quad (q; q)_\infty
\]
and
\[
H(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q; q)_\infty} = \frac{f(-q^5)}{f(-q^3; -q^3)} = \frac{1}{(q; q^5)_\infty (q^3; q^5)_\infty (q^5; q^5)_\infty} \quad (q; q)_\infty ,
\]
where the functions \( f(a, b) \) and \( f(-q) \) are given as in (9) and (14), respectively. For a detailed historical account of the Rogers-Ramanujan continued fraction (16) and identities (17) and (18), one can see [5, p. 77].

The following continued fractions were recalled (see, e.g., [6]):
\[
(q^2; q^2)_\infty (-q; q)_\infty = \frac{(q; q^2)_\infty (q^2; q^2)_\infty}{(q; q^2)_\infty} = \frac{1}{1 -} \frac{q}{1 +} \frac{q(1 - q)}{1 -} \frac{q^2(1 - q^2)}{1 +} \frac{q^5(1 - q^3)}{1 -} \cdots ;
\]
\[
(q; q^5)_\infty (q^4; q^5)_\infty = \frac{1}{1 +} \frac{q^2}{1 +} \frac{q^3}{1 +} \frac{q^4}{1 +} \frac{q^5}{1 +} \frac{q^6}{1 +} \cdots ;
\]
\[
C(q) = \frac{(q^2; q^5)_\infty (q^3; q^5)_\infty}{(q; q^5)_\infty (q^4; q^5)_\infty} = 1 + \frac{q}{1 +} \frac{q^2}{1 +} \frac{q^3}{1 +} \frac{q^4}{1 +} \frac{q^5}{1 +} \frac{q^6}{1 +} \cdots .
\]
Very recently, Andrews et al. [7] investigated combinatorial partition identities associated with the following general family:

\[ R(s, t, l, u, v, w) := \sum_{n=0}^{\infty} q^{s(n^2)+tn} r(l, u, v; w; n), \]  

(22)

where

\[ r(l, u, v; w: n) := \sum_{j=0}^{\lfloor \frac{n}{w} \rfloor} (-1)^j \frac{q^{uv(j^2)+(w-ul)j}}{(q; q)_{n-wj}(q^{uv}; q^{uv})_j}. \]  

(23)

We recall the following combinatorial partition identities [7, p. 106, Theorem 3]:

\[ R(2, 1, 1, 2, 2) = (-q; q^2)_{\infty}; \]  

(24)

\[ R(2, 2, 1, 1, 2, 2) = (-q^2; q^2)_{\infty}; \]  

(25)

\[ R(m, m, 1, 1, 2) = \frac{(q^{2m}; q^{2m})_{\infty}}{(q^m; q^{2m})_{\infty}}. \]  

(26)

Here, in this paper, by combining the above-mentioned results and using some easily-derivable identities for \( q \)-products, we aim to establish certain (presumably) new identities which reveal interrelationships between 3-dissection property, continued-fraction identities and combinatorial partition identities.

\section{Main Results}

For our purpose, we begin by recalling some known identities (see [4]) which are given as in the following lemma.

\textbf{Lemma 2.1.} Each of the following relationships holds true:

\[ A(q) = A(q^3) + 2qC(q^3); \]  

(27)

\[ B(q) = A(q^3) - qC(q^3); \]  

(28)

\[ C(q) = \frac{3((q^3; q^3)_{\infty})^3}{(q; q)_{\infty}}; \]  

(29)

\[ A(q)A(q^2) = B(q)B(q^2) + qC(q)C(q^2), \]  

(30)

where \( A(q), B(q) \) and \( C(q) \) are cubic theta functions given by

\[ A(q) = \sum_{m, n=-\infty}^{\infty} q^{m^2+mn+n^2}; \]  

(31)
holds true:

\[ B(q) = \sum_{m,n=-\infty}^{\infty} \omega^{m-n} q^{m^2+mn+n^2} \left( \omega = \exp \left( \frac{2\pi i}{3} \right) \right); \quad (32) \]

\[ C(q) = \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2+m+n}. \quad (33) \]

Our main results are given in the following theorem.

**Theorem 2.2.** Each of the following interrelationships between 3-dissection property, combinatorial partition identities and continued-fraction identities holds true:

\[
\frac{3}{(q; q)_\infty (q^2; q^2)_\infty} = \frac{A(q^6)C(q^3)}{\{(q^3; q^3)_\infty \}^3 \{(q^6; q^6)_\infty \}^3} + \frac{qA(q^3)C(q^6)}{q^2C(q^3)C(q^6)} + \frac{qA(q^3)C(q^6)}{\{(q^3; q^3)_\infty \}^3 \{(q^6; q^6)_\infty \}^3};
\]

\[
A(q^6)C(q^3) + qA(q^3)C(q^6) + q^2C(q^3)C(q^6)
= 3 \left( \frac{\{(q^3; q^3)_\infty \}^3 \{(q^6; q^6)_\infty \}^4 \{(q^6; q^{12})_\infty \}^5}{(q^2; q^2)_\infty \{(q^2; q^4; q^6; q^{10}; q^{12})_\infty \}^3} \right) \times \left( \frac{1}{1+ \frac{1}{1+ \frac{1}{1+ \cdots}}} \right)^2 \left( \frac{1}{1+ \frac{1}{1+ \frac{1}{1+ \cdots}}} \right)^2; \quad (34)
\]

\[
A(q^6)C(q^3) + qA(q^3)C(q^6) + q_2C(q^3)C(q^6)
= 3 \left( \frac{\{(q^3; q^3)_\infty \}^3 \{(q^6; q^6)_\infty \}^4 \{(q^6; q^{12})_\infty \}^5 \{(q^{12}; q^{24}; q^{36}; q^{48}; q^{60})_\infty \}^2}{(q^2; q^2)_\infty \{(q^2; q^4; q^6; q^{10}; q^{12})_\infty \}^3 \{(q^6; q^{18}; q^{24}; q^{36}; q^{54}; q^{60})_\infty \}^2} \right) \times \left( \frac{1}{1+ \frac{1}{1+ \frac{1}{1+ \cdots}}} \right)^2; \quad (35)
\]

\[
A(q^6)C(q^3) + qA(q^3)C(q^6) + q^2C(q^3)C(q^6) = 3 \left( R(30, 30, 1, 1, 1, 2) \right)^2 \times \left( \frac{\{(q^3; q^3)_\infty \}^3 \{(q^6; q^6)_\infty \}^4 \{(q^6; q^{12})_\infty \}^2 \{(q^{12}; q^{24}; q^{36}; q^{48}; q^{60})_\infty \}^2}{(q^2; q^2)_\infty \{(q^2; q^4; q^6; q^{10}; q^{12})_\infty \}^3 \{(q^6; q^{18}; q^{24}; q^{36}; q^{54}; q^{60})_\infty \}^2} \right); \quad (36)
\]

**Proof of (34):** From (29), after a suitable manipulation, we have

\[
\frac{1}{(q; q)_\infty (q^2; q^2)_\infty} = \frac{C(q)C(q^2)}{9 \{(q^3; q^3)_\infty \}^3 \{(q^6; q^6)_\infty \}^3}; \quad (38)
\]
Similarly, from (27) and (28), we get, respectively, the following identities:

\[ A(q)A(q^2) = A(q^3)A(q^6) + 2q^2A(q^3)C(q^6) + 2qA(q^6)C(q^3) + 4q^3C(q^3)C(q^6) \]

(39)

and

\[ B(q)B(q^2) = A(q^3)A(q^6) - q^2A(q^3)C(q^6) - qA(q^6)C(q^3) + q^3C(q^3)C(q^6). \]

(40)

Combining the identities in (30), (39) and (40), we obtain

\[ C(q)C(q^2) = 3A(q^6)C(q^3) + 3qA(q^3)C(q^6) + 3q^2C(q^3)C(q^6). \]

(41)

Substituting \( C(q)C(q^2) \) in (41) for the \( C(q)C(q^2) \) in (38) is seen to yield the desired identity (34).

**Proof of (35):** A rearrangement of the result in (38) gives

\[ A(q^6)C(q^3) + qA(q^3)C(q^6) + q^2C(q^3)C(q^6) = 3 \left( \frac{(q^3; q^3)_\infty^3 (q^6; q^6)_\infty^3}{(q; q)_\infty (q^2; q^2)_\infty} \right). \]

(42)

Using some properties of \( q \)-product identities, after a simplification, we get

\[
A(q^6)C(q^3) + qA(q^3)C(q^6) + q^2C(q^3)C(q^6) \\
= 3 \left( \frac{(q^3; q^3)_\infty^3 (q^6; q^6)_\infty^4}{(q; q^2)_\infty (q^2; q^4, q^8, q^{10}, q^{12})_\infty^2} \right) \\
\times \left( \frac{(q^{12}; q^{60})_\infty (q^{48}; q^{60})_\infty}{(q^{24}; q^{60})_\infty (q^{36}; q^{60})_\infty} \right)^2 \frac{(q^{24}; q^{60})_\infty (q^{36}; q^{60})_\infty}{(q^{12}; q^{60})_\infty (q^{48}; q^{60})_\infty}. \]

(43)

Replacing \( q \) by \( q^{12} \) in (20) and (21), and applying the resulting identities to (44), we are led to the desired identity (35).

**Proof of (36):** Applying some suitable properties of \( q \)-product identities to (34), we obtain

\[
A(q^6)C(q^3) + qA(q^3)C(q^6) + q^2C(q^3)C(q^6) \\
= 3 \left( \frac{(q^3; q^3)_\infty^3 (q^6; q^6)_\infty^4}{(q; q^2)_\infty (q^2; q^4, q^8, q^{10}, q^{12})_\infty^2} \right) \frac{(q^{12}; q^{60})_\infty (q^{48}; q^{60})_\infty}{(q^{30}; q^{60})_\infty} \right)^2 \frac{(q^{24}; q^{60})_\infty (q^{36}; q^{60})_\infty}{(q^{12}; q^{60})_\infty (q^{48}; q^{60})_\infty}. \]

(44)

which, upon using (19), is seen to yield the identity (36).

**Proof of (37):** Applying the result of \( m = 30 \) in (26) to (44) yields (37).

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