Qualitative Analysis of a Model for
the Classic Dengue Dynamics

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Abstract

A mathematical model for a non-linear continuous stochastic process is presented. Analysis of the stability by uncoupling system is realized.

Keywords: Continuous stochastic process, mathematical model, dengue dynamics, Aedes aegypti

1 Introduction

Dengue is a viral disease which is transmitted to humans by vector mosquitoes of the Aedes genus, being aegypti the most important one in the disease transmission. Dengue fever is considered as a disease with a greatest epidemiological impact [1],[2].
There are several models that guide the study of the transmission dynamics of the classic dengue with constant and variable population [3], vertical transmission in the mosquito [4], spatial-temporal dynamics [5], dynamics of different virus serotypes [6], population structure [7], stochastic dynamics [8], mosquito control by adulticides, larvicides, and constant mortality rate [9], and applying the Pontryagin’s Maximum Principle [10].

Unlike a model of non-linear continuous stochastic process with discrete stages and variable intensities of Poisson flows. Through deterministic model to the average magnitudes [11], the following considerations are proposed: Constant size human population $N$ with constant birth rate $\mu$ equal to its natural mortality rate $\mu$. In the model, $\alpha \psi_1 \left( \frac{m_i}{m} \right) x$ are the incidences in the human population, $\beta \psi_2 \left( \frac{y}{N} \right) m_s$ the mosquito population and $\psi_3 \left( \frac{a}{I} \right)$ the saturation function of the immature stage.

Moreover, $x, y, z$ and $N$ are the average of susceptible, infectious, immune, and total population at time $t$, respectively. $a$ : average of immature stages (egg, larvae, pupa) of aquatic phase at time $t$, $I$: charge capacity of the breeding places, $\alpha$ and $\beta$ are inverse magnitudes of the time unit, $\theta$ : is the infected people recovery rate, $\phi$ : number of mosquito eggs per day, $\epsilon$ : natural mortality of susceptible and infectious mosquitoes, $\pi$ : natural mortality of the immature stages (egg, larvae, pupa), $\omega^{-1}$ : average time of development, from egg to mature stage.

The differential equations of the dynamics are:

\[
\begin{align*}
\frac{dx}{dt} &= \mu N - \alpha \psi_1 \left( \frac{m_i}{m} \right) x - \mu x \\
\frac{dy}{dt} &= \alpha \psi_1 \left( \frac{m_i}{m} \right) x - (\mu + \theta) y \\
\frac{dz}{dt} &= \theta y - \mu z \\
\frac{dm_s}{dt} &= \omega a - \beta \psi_2 \left( \frac{y}{N} \right) m_s - \epsilon m_s \\
\frac{dm_i}{dt} &= \beta \psi_2 \left( \frac{y}{N} \right) m_s - \epsilon m_i \\
\frac{da}{dt} &= \phi m \psi_3 \left( \frac{a}{I} \right) - (\pi + \omega) a
\end{align*}
\]

where $\psi_i, i = 1, 2$ are sigmoidal functions and $\psi_3$ is antisigmoidal, these functions are continuous and satisfy the following conditions:
2 Main Results

We know that $z = N - x - y$, $m = m_s + m_i$ and adding the equations (4) and (5), the previous system is reduced and decoupled in two subsystems, A and B.

\[
(A) \begin{cases} 
  \frac{dx}{dt} = \mu N - \alpha \psi_1 \left( \frac{m_i}{m} \right) x - \mu x \\
  \frac{dy}{dt} = \alpha \psi_1 \left( \frac{m_i}{m} \right) x - (\mu + \theta) y \\
  \frac{dm_i}{dt} = \beta \psi_2 \left( \frac{y}{N} \right) (m - m_i) - \epsilon m_i 
\end{cases}
\]

\[
(B) \begin{cases} 
  \frac{dm}{dt} = \omega a - \epsilon m \equiv F(m, a) \\
  \frac{da}{dt} = \phi m \psi_3 \left( \frac{a}{I} \right) - (\pi + \omega) a \equiv G(m, a)
\end{cases}
\]

We suppose that the initial conditions belong to the Σ region of the phase space, that is the zone with biological sense, namely.

\[
\Sigma = \{(x, y, m_i, m, a) : x \geq 0, y \geq 0, m_i \geq 0, m \geq 0, m_i \leq m, 0 \leq a \leq I\}
\]
The biological sense region of the subsystem B is defined as:

\[ \Sigma_B = \{(m, a) : m \geq 0, 0 \leq a \leq I\} \]

and the growing threshold of the mosquito \( R_M = \frac{\phi \omega}{\epsilon (\pi + \omega)} \), \( R_M \) indicates the average number of adult mosquitoes produced per mosquito, where \( \frac{\phi}{\epsilon} \) is the average number of oviposited eggs per female mosquito, and \( \frac{\omega}{\pi + \omega} \) is the transformation probability to adult stage.

The subsystem (B) has two stationary solutions with biological sense: the trivial one in the case when \( \psi_3(0) < \frac{1}{R_M} < 1 \), and the coexistence solution that satisfy \( m = \omega a \), the equation \( \psi_3(\frac{a}{I}) = \frac{1}{R_M} \), and it is related to the mosquito growing threshold for \( R_M > 0 \).

The local stability analysis of the subsystem (B) based on a Jacobian matrix is realized.

\[
\begin{pmatrix}
-\epsilon & -\phi \psi_3' \left( \frac{a}{I} \right) \\
\phi \psi_3' \left( \frac{a}{I} \right) & \phi \hat{m} \psi_3' \left( \frac{a}{I} \right) - (\pi + \omega)
\end{pmatrix}
\]

and its corresponding characteristic equation.

\[
\lambda^2 + \left[ \epsilon + \pi + \omega + \frac{\phi}{I} \hat{m} \psi_3' \left( \frac{a}{I} \right) \right] \lambda - \frac{\epsilon \phi}{I} \hat{m} \psi_3' \left( \frac{a}{I} \right) + \epsilon (\pi + \omega) \left( 1 - R_M \psi_3 \left( \frac{a}{I} \right) \right) = 0 \tag{9}
\]

which has roots \( \lambda_{\pm} = \frac{1}{2}(\zeta \pm \sqrt{\xi}) \), where \( \zeta = -\left( \epsilon + \pi + \omega - \frac{\phi}{I} \hat{m} \psi_3' \left( \frac{a}{I} \right) \right) \), \( \chi = -\frac{\phi}{I} \hat{m} \psi_3' \left( \frac{a}{I} \right) \), \( \xi = \zeta^2 - 4\chi \) and \( \psi_3' \) denotes the derivative of the function \( \psi_3 \).

Now, we analyze different stability cases:

Case \( \frac{1}{R_M} > \psi_3(0) \): It accomplishes that \( 1 - R_M \psi_3(0) > 0 \), therefore, \( \chi > 0 \) and \( 0 < \xi < \zeta^2 \), then the stationary solution \((0,0)\) is a stable node and it is a trivial solution of the non-linear local asymptotically-stable system.

Case \( \frac{1}{R_M} < \psi_3(0) \) : \( \chi < 0 \), therefore, the trivial solution is unstable (saddle-like) according to Lyapunov. For the stability of the coexistence stationary solution \((\hat{m}, \hat{a})\), it is obtained \( \frac{1}{R_M} < \psi_3(0) \) :

\[
\zeta < 0 \quad , \quad \xi = \left[ \epsilon + \pi + \omega - \frac{\phi}{I} \hat{m} \psi_3' \left( \frac{a}{I} \right) \right]^2 + 4\epsilon (\pi + \omega) > 0
\]

Equation (9) has roots \( \lambda_- < 0 \) and \( \lambda_+ < 0 \) from where we conclude that the coexistence solution \((\hat{m}, \hat{a})\) is local and asymptotically stable. The stability results are summarized in the following theorem.
Theorem 2.1 For all $R_M > 0$ the subsystem (B) admits the trivial solution, $R_M \psi_3(0) > 1$ allows the coexistence stationary solution of mature and immature stages. The trivial stationary solution of the subsystem (B) is local and asymptotically stable as $R_M \psi_3(0) < 1$ (being a node). If $R_M \psi_3(0) > 1$ is unstable (saddle-like). When $R_M \psi_3(0) > 1$ then the coexistence stationary solution $(\hat{m}, \hat{a})$ is local and asymptotically stable (also being a node).

For the case $\frac{1}{R_M} = \psi_3(0)$ we talk about stability with biological sense of the trivial solution but it is not possible to demonstrate it. In general, the trivial stability solution is in a Lyapunov sense, except in the particular case in which $\psi_3 = 1$ over all the interval $(-\infty, 0]$.

2.1 Periodic solutions

When $\frac{1}{R_M} = \psi_3(0)$ the characteristic equation (9) corresponding to the trivial stationary solution presents a null root, for this reason we study the existence or nonexistence of periodic orbits through the application of the Bendixon-Dulac theorem [12].

In fact, considering $F(m, a)$ and $G(m, a)$ as in (8), it is easy to see that

$$\frac{\partial F}{\partial m} + \frac{\partial G}{\partial a} = -(\epsilon + \pi + \omega) + \frac{\phi}{I} m \psi_3' \left( \frac{\hat{a}}{I} \right)$$

Using $\psi_3' \left( \frac{\hat{a}}{I} \right) \leq 0$ and if we suppose that $m \geq 0$, then we deduce that $\frac{\partial F}{\partial m} + \frac{\partial G}{\partial a}$ is negative. This fact allow us to conclude the following theorem.

Theorem 2.2 For all $R_M > 0$ the subsystem (B) does not present periodic orbits in the biological sense region $\Sigma_B$.

2.2 Analysis of the subsystem (A)

Substituting the stable stationary solutions $(0, 0)$ and $(\hat{m}, \hat{a})$ of the subsystem (B) in (A), the stationary solution $(N, 0, 0)$ without infected people nor mosquitoes carrying virus has a characteristic equation with a negative real part, where the stationary solution $(N, 0, 0)$ is local and asymptotically stable for all $R_M > 0$. Moreover, the stationary solution $(N, 0, 0, 0, 0)$ of the system (7) and (8) is local and asymptotically stable if $R_M \psi_3(0) < 1$.

For the case $R_M \psi_3(0) > 1$, substituting the stable stationary solution $(\hat{m}, \hat{a})$ in the subsystem (A) equations, the following system is obtained.
\[
\begin{align*}
\frac{dx}{dt} &= \mu N - \alpha \psi_1 \left( \frac{m_i}{m} \right) x - \mu x \\
\frac{dy}{dt} &= \alpha \psi_1 \left( \frac{m_i}{m} \right) x - (\mu + \theta) y \\
\frac{dm_i}{dt} &= \beta \psi_2 \left( \frac{y}{N} \right) (\hat{m} - m_i) - \epsilon m_i
\end{align*}
\] (10) (11) (12)

We have analyzed the stationary solutions of the system (10)-(12) using \( \frac{dx}{dt} = 0 \), \( \frac{dy}{dt} = 0 \) and \( \frac{dm_i}{dt} = 0 \), the system is solved dividing (12) into \( \hat{m} \) and using the following change of variables \( X = x, Y = \frac{y}{N} \), and \( Z = \frac{m_i}{m} \), to get the equivalent system:

\[
X = N \left( 1 - \frac{\mu + \theta}{\mu} Y \right), \quad \psi_1(Z) = \frac{(\mu + \theta)Y}{\alpha \left( 1 - \frac{\mu + \theta}{\mu} Y \right)}, \quad \psi_2(Y) = \frac{\epsilon Z}{\beta(1 - Z)}
\] (13)

So that, \( \hat{x}, \hat{y} \) and \( \hat{m}_i \) have epidemiological sense if they meet the following conditions \( \hat{X} \geq 0, \ 0 \leq \hat{Y} \leq 1, \ 0 \leq \hat{Z} \leq 1, \ \hat{X} + \hat{N}\hat{Y} \leq N \).

**Theorem 2.3** System (10)-(12) has a unique solution with epidemiological sense.

**Proof 1.** From the third equation in (13) we have

\[
Z = \frac{\beta \psi_2(Y)}{\beta \psi_2(Y) + \epsilon}
\] (14)

Substituting (14) in the second equation of (13) we obtain

\[
F(Y) = \psi_1 \left( \frac{\beta \psi_2(Y)}{\beta \psi_2(Y) + \epsilon} \right) = \frac{\mu Y}{\alpha \left( \frac{\mu}{\mu + \theta} - Y \right)}
\] (15)

It is easy to see that \( \epsilon \beta \psi_1' \left( \frac{\beta \psi_2(Y)}{\beta \psi_2(Y) + \epsilon} \right) \frac{\psi_1'(Y)}{\beta \psi_2(Y) + \epsilon} \) is strictly positive, moreover,

\[
\lim_{Y \to -\infty} F(Y) = \psi_1(0), \quad \lim_{Y \to +\infty} F(Y) = \psi_1 \left( \frac{\beta}{\beta + \epsilon} \right).
\]

Where \( F \) has a sigmoidal behavior. The functions \( F(Y) \) and \( \frac{Y}{\alpha \left( \frac{\mu}{\mu + \theta} - Y \right)} \) have a unique root in \( 0 < \hat{Y} < \frac{\mu}{\mu + \theta} \).

Using \( \hat{Y} \) in (15) and the first equation of (13) to achieve:

\[
0 < \hat{Z} = \frac{\beta \psi_2(Y)}{\beta \psi_2(Y) + \epsilon} < 1, \quad 0 < \hat{x} = N \left( 1 - \frac{\mu + \theta}{\mu} \hat{Y} \right)
\] (16)

Clearly there is a unique solution \( (\hat{x}, \hat{y}, \hat{m}_i) \) of the system (13) as well as \( \hat{y} = N\hat{Y} \), where \( \hat{Y} \) is the only root of the equation (16), furthermore \( \hat{x} = \hat{X}, \hat{m}_i = \hat{m}\hat{Z} \), where \( \hat{X} \) and \( \hat{Z} \) are given by (16).
The Jacobian-like matrix of the equation system (10)-(12) evaluated in the stationary solution \((\hat{x}, \hat{y}, \hat{m}_i)\) is

\[
\begin{pmatrix}
-\alpha \psi_1(\hat{Z}) - \mu & 0 & -\frac{\hat{X}}{m_i} \psi_1'(\hat{Z}) \\
-\mu & -(\theta + \mu) & 0 \\
0 & \beta \frac{\hat{m}_i}{N} (1 - \hat{Z}) \psi_2'(\hat{Y}) & -[\beta \psi_2(\hat{Y}) + \epsilon]
\end{pmatrix}
\]

whose characteristic equation is

\[P(\lambda) = \lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3 = 0\]

where

\[a_1 = \alpha \psi_1(\hat{Z}) + \beta \psi_2(\hat{Y}) + 2\mu + \theta + \epsilon\]

\[a_2 = (\alpha \alpha \psi_1(\hat{Z}) + \mu)(\mu + \theta) + (\alpha \alpha \psi_1(\hat{Z}) + 2\mu + \theta)(\beta \psi_2(\hat{Y}) + \epsilon)\]

\[a_3 = (\alpha \alpha \psi_1(\hat{Z}) + \mu)(\mu + \theta)(\beta \psi_2(\hat{Y}) + \epsilon) - \mu \beta \frac{\hat{X}}{N} (1 - \hat{Z}) \psi_1'(\hat{Z}) \psi_2'(\hat{Y})\]

\[= (\theta + \mu)[\beta \psi_2(\hat{Y})(\alpha \psi_1(\hat{Z}) + \mu) + \alpha \epsilon \psi_1(\hat{Z})] + \mu \alpha \beta \frac{\hat{X}}{N} \hat{Z} \psi_1'(\hat{Z}) \psi_2'(\hat{Y}) + \mu \epsilon (\theta + \mu)(1 - R_0)\]

with

\[R_0 = \frac{\alpha \beta}{\epsilon (\theta + \mu)} \psi_1'(\hat{Z}) \psi_2'(\hat{Y}) \frac{\hat{X}}{N}\]

the Basic Reproductive Number, defined as the average number of secondary cases produced by a single-infected during his/her infectious period in a completely susceptible population.

The stability analysis of the stationary solution \((\hat{x}, \hat{y}, \hat{m}_i)\) defines the sign of the roots of \(P(\lambda) = 0\) in terms of \(R_0\).

### 3 Conclusions

In this paper we model the transmission dynamics and classic dengue incidence as a non-linear continuous stochastic process using a differential equations system to the average variables. We have done a qualitative analysis of the model by uncoupling the system. The incorporation of incidence functions for infected people, mosquitoes carrying virus and the charge capacity of the breeding places with its characterization allow us to propose explicit forms according to the real behavior of the infection dynamics. The methodology to uncouple the system and the stability analysis of the general system into subsystems is applicable to differential equation systems that model the transmission dynamics and incidence of diseases like: malaria, chagas, leishmaniasis, virus del este del Nilo, sexually transmitted diseases, among others.

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