Analysis of a Commodity Market Model with Delay

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Abstract
A discrete model of commodity market is converted into a continuous model augmented with time delay. Taking the delay as a bifurcation parameter, the system loses its stability and a Hopf bifurcation occurs when the delay passes through some critical values.

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1 Introduction
In recent years, the interest of researchers in economic systems with time delays has become increasingly strong (see e.g. [1]-[7]). In particular, the use of time delay was used by Matsumoto and Szidarovszky [8] to obtain continuous-time models via discrete-time models. In this paper, following Matsumoto and Szidarovszky [8], the discrete behavioral commodity market model, with consumers, producers and heterogeneous speculators, developed in He and Westerhoff [9] is converted into a continuous model augmented with time delay $\tau \geq 0$. The modified system happens to be described by a delay differential equation

$$\dot{S} = -S + S_\tau + \left[ m(F - S_\tau) - b \frac{(F - S_\tau)}{1 + d(F - S_\tau)^2} + c \frac{d(F - S_\tau)^3}{1 + d(F - S_\tau)^2} \right],$$

where $b, c, d, m$ are positive constants, $F$ is the equilibrium price and $S_\tau = S(t - \tau)$. The parameter $a$ that appears in He and Westerhoff [9] has been
taken equal to one since it can be treated as a scaling factor. Henceforth, for simplicity, it is assumed \( b \neq m \). We will study the stability and local Hopf bifurcation for the system (1). Our main result is that as the delay increases, the positive equilibrium loses its stability and a sequence of Hopf bifurcations occur at the positive equilibrium, i.e. a family of periodic orbits bifurcates from the positive equilibrium.

2 Stability and bifurcation analysis

In this section, we shall focus on analyzing the corresponding linearized system at the positive equilibrium of the Eq. (1) and investigate the stability of this equilibrium point and the existence of local Hopf bifurcations occurring at the positive equilibrium.

Lemma 2.1. Eq. (1) has the following equilibria: \( S_* = F \) and \( S_* = S_\pm \) if \( b > m \), where

\[
S_\pm = F \pm \sqrt{\frac{b-m}{(m+c)d}}.
\]

Proof. By setting \( \dot{S} = 0 \) and \( S_\tau = S \) for all \( t \) we find

\[
(S - F) \left[(m+c) d(F-S)^2 + m - b\right] = 0
\]

Obviously, \( S = F \) is always a steady state. For \( b > m \), we have

\[
(F-S)^2 = \frac{b-m}{(m+c)d} \implies |F-S| = \sqrt{\frac{b-m}{(m+c)d}} \implies F - S = \pm \sqrt{\frac{b-m}{(m+c)d}}.
\]

Hence, the statement. \( \square \)

The linearization of (1) at \( S_* \) is given by

\[
\dot{S} = -(S-S_*) + M (S_\tau - S_*),
\]

where

\[
M = 1 - (m+c) + \frac{b+c}{1 + d(F - S_*)^2} - \frac{2d(b+c)(F - S_*^2)}{\left[1 + d(F - S_*)^2\right]^2},
\]

whose associated characteristic equation is

\[
\lambda = -1 + Me^{-\lambda\tau}
\]

Remark 2.2. \( M = 1 + b - m \) if \( S_* = F \), \( M = 1 - 2d(b+c)(F - S_*^2)/[1 + d(F - S_*^2)]^2 = 1 - 2(b-m)(m+c)/(b+c) \) if \( S_* = S_\pm \).
Lemma 2.3. Let $\tau = 0$. If $b < m$, then the equilibrium $S_* = F$ is stable. If $b > m$, then the equilibrium $S_* = F$ is unstable and the equilibria $S_* = S_\pm$ are stable.

Proof. In case there is no delay, Eq. (3) becomes $\lambda = -1 + M$. If $S_* = F$, then $\lambda = b - m$. Thus, $\lambda < 0$ for $b < m$ and $\lambda > 0$ for $b > m$. If $S_* = S_\pm$, then $\lambda = -2(b - m)(m + c)/(b + c) < 0$. The conclusion holds. \hfill \Box

As $\tau$ increases, the stability of the equilibrium point $S_*$ will change if (3) has zero or a pair of purely imaginary eigenvalues. It is immediate that the case $\lambda = 0$ is not possible, since otherwise it would give the contradiction $b = m$ if $S_* = F$ and $F = S_\pm$ if $S_* = S_\pm$.

Next, we look for the existence of a root $\lambda = i\omega$ for (3). Without loss of generality, since the complex roots of (3) appear as complex conjugate pairs, we assume $\lambda = i\omega$ ($\omega > 0$) to be a root of (3). Substituting in (3), and separating the real and imaginary parts of this equation, we obtain

$$
\omega = -M \sin \omega \tau, \quad 1 = M \cos \omega \tau. \quad (4)
$$

Eliminating $\tau$ from (4) yields

$$
\omega^2 = M^2 - 1.
$$

If $S_* = F$, then $\omega^2 = (b-m)(b-m+2)$. Hence, $\omega > 0$ if $b-m > 0$ or $b-m < -2$. If $S_* = S_\pm$, then $\omega^2 = 4(b-m)(m+c)[(b-m)(m+c) - (b+c)]/(b+c)^2$. Therefore, $\omega > 0$ if $(b-m)(m+c) - (b+c) > 0$. We easily obtain the following results about the equilibrium points $S_*$.

Proposition 2.4. If $b-m < -2$ or $b-m > 0$ or $b-m > 0$ and $(b-m)(m+c) - (b+c) > 0$, then Eq. (3) has pair of purely imaginary roots $\lambda = \pm i\omega_0$ at a sequence of critical values $\tau_j$, $j \in \mathbb{N}^0 = \mathbb{N} \cup \{0\}$, where

$$
\omega_0 = \sqrt{M^2 - 1} \quad \text{and} \quad \tau_j = \frac{1}{\omega_0} \left[ \arctan (-\omega_0) + (2j + 1) \pi \right]. \quad (5)
$$

We now check the transversality condition for Hopf bifurcation.

Lemma 2.5. For $\tau = \tau_j$, $j \in \mathbb{N}^0$, $\xi = \pm i\omega_0$ are simple roots of (3) and

$$
\frac{d(\text{Re}\lambda)}{d\tau} \bigg|_{\tau=\tau_j} > 0.
$$

Proof. A direct calculation shows that $\lambda = i\omega_0$ is a simple root of (3). Let $\xi(\tau) = \alpha(\tau) + i\omega(\tau)$ be a root of (3) satisfying $\alpha(\tau_j) = 0$ and $\omega(\tau_j) = \omega_0$. Differentiating both sides of (3) with respect $\tau$ gives that

$$
\left( \frac{d\lambda}{d\tau} \right)^{-1} = -\frac{1}{(\lambda + 1)\lambda} - \frac{\tau}{\lambda}.
$$
Therefore,
\[
\text{sign} \left\{ \frac{d(\text{Re}\lambda)}{d\tau} \bigg|_{\tau=\tau_j} \right\} = \text{sign} \left\{ \text{Re} \left( \frac{d\lambda}{d\tau} \right)^{-1} \bigg|_{\tau=\tau_j} \right\} = \text{sign} \left\{ \frac{1}{\omega_0^2 + 1} \right\}.
\]

This implies the statement. \(\square\)

The above result implies that the pair of pure imaginary roots crosses the imaginary axis from the left to the right as \(\tau\) continuously varies from a number less than \(\tau_j\) to one greater than \(\tau_j\).

Summarizing our analysis, we have the following conclusions.

**Theorem 2.6.** Let \(\omega_0, \tau_j, j \in \mathbb{N}^0\), be defined as in (5). If \(b - m < -2\) (resp. \(b > m\) and \((b - m)(m + c) - (b + c) > 0\)), then the equilibrium \(S_* = F\) (resp. the equilibria \(S_* = S_{\pm}\)) of (1) is locally asymptotically stable for \(\tau \in [0, \tau_0)\) and unstable for \(\tau > \tau_0\). Furthermore, Eq. (1) undergoes a Hopf bifurcation at the equilibrium \(F\) (resp. \(S_{\pm}\)) when \(\tau = \tau_j, j \in \mathbb{N}^0\).

**References**


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