Convolution of Generated Random Variable from Exponential Distribution with Stabilizer Constant

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Abstract

The new distribution is constructed by generating random variables from an exponential distribution with stabilizer constant. It is presented the convolution from this identically and independent new random variable by using analytical methods in the form of probability density function and cumulative distribution function. In addition, it is presented some properties of the exponential distribution with stabilizer constant and its convolution including mean, variance, moment generating function, characteristic function, skewness and kurtosis.

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1 Introduction

The exponential distribution is a continuous probability distribution with memoryless property and measured failure rates. It makes this distribution widely used in problems of reliability such as to show the solution of queue with service time as the sum of exponential random variables. The distribution of the sum of independent random variables in this sense is called convolution and the statistical methods to obtain the distribution have been introduced in Feller [5] by using its definition or properties of moment generating function. The distribution of convo-
olution from exponential random variable has been discussed by Akkouchi [3] where the probability distribution function is obtained by using moment generating function approach with induction and analytical methods. Therefore convolution of the exponential distribution with same parameters is obviously obtained by using characteristic function and its property of infinite divisibility becoming gamma distribution.

The research on convolution of exponential distribution and its properties has attracted many researchers because of its applications in other fields, such as Jasiulewicz and Kordecki [1] have shown convolution of Erlang and Pascal distribution as an exponential distribution family in reliability. Meanwhile, The analytical scheme to obtain convolution of exponential distribution has shown by Ma and Liu [4], while Smaili et al. [2] introduced hypoexponential distribution as the convolution of the exponential distribution with distinct parameters by using the Laplace transform properties.

The convolutions related to exponential distributions have introduced by these previous papers developed based on established distributions, then it is interesting to show convolution in another way, such as by constructing the new probability distribution of its generated random variables. In this article we derived convolution from generating random variables from an exponential distribution with stabilizer constant. The process of construction this new probability density function and its properties is exposed in Section 2 and furthermore its convolution and properties in Section 3.

2 Generated Random Variable from Exponential Distribution

Let \( X \) be random variable from an exponential distribution with parameter \( \lambda \). Probability density function of this random variable is given by

\[
f(x; \lambda) = \lambda e^{-\lambda x}
\]

for positive \( \lambda \) and all positive real numbers \( x \). We denote \( W \) as random variable related to \( x \) which way generated from the exponential distribution with continuous values spread in the form

\[
w_i = \frac{x_i}{x_m}
\]

where it is defined \( x_m = \max\{x_1, x_2, ..., x_n\} \) for \( x_i \in (0, x_m] \) and \( n, m \in Z^+ \). Then random variable \( W \) has probability density function and cumulative distribution function respectively as follows

\[
f(w; \lambda) = \theta \lambda e^{-\lambda w},
\]

\[
F((w; \lambda)) = 1 - \theta e^{-\lambda w},
\]
for positive $\lambda$ and $0 < w \leq 1$ with $\theta$ as stabilizer constant to maintain the nature of probability density function. The stabilizer constant in this new introduced distribution is obtained as

$$\theta = \frac{1}{1 - e^{-k}}.$$ 

The properties of this distribution are given in expression of mean, variance, moment generating function and characteristic function respectively as follows

$$\mu_w = \frac{1}{\lambda} - \theta e^{-\lambda},$$

$$\sigma_w^2 = \theta \left( \frac{2 - 2e^{-\lambda}}{\lambda^2} - \frac{2e^{-\lambda}}{\lambda} - e^{-\lambda} \right) - \theta^2 \left( \frac{1 + e^{-2\lambda} - 2e^{-\lambda}}{\lambda^2} + \frac{2e^{-2\lambda} - 2e^{-\lambda}}{\lambda} + e^{-2\lambda} \right),$$

$$M_w = \frac{-\theta \lambda}{\lambda - t} (e^{-(\lambda-t)} - 1) \text{ for } -\infty < t < \infty,$$

$$\phi_w = \frac{-\theta \lambda}{\lambda - it} (e^{-(\lambda-it)} - 1) \text{ for } -\infty < t < \infty.$$ 

The most informative properties to express the shape of this distribution are skewness and kurtosis respectively in the following forms

$$Skew(W) = \frac{sp_1(\theta, \lambda)(sp_2(\theta, \lambda) - sp_3(\theta, \lambda))}{(\theta^2 e^{2\lambda} (2 - \theta) + (1 + \lambda)^2 + \theta e^\lambda (2 + \theta + \lambda^2 - 2\theta(1 + \lambda)))^{3/2}},$$

$$Kurt(W) = \frac{e^{-\lambda} (kp_1(\theta, \lambda) - kp_2(\theta, \lambda) - kp_1(\theta, \lambda) - kp_3(\theta, \lambda) - kp_3(\theta, \lambda))}{\theta (\theta^2 e^{2\lambda} (2 - \theta) - (1 + \lambda)^2 - \theta e^\lambda (2 + 2\lambda + \lambda^2 - 2\theta(1 + \lambda)))^2},$$

where functions in the form of skewness and kurtosis are defined as follows

$$sp_1(\theta, \lambda) = \theta^2 e^{\lambda} ((\theta^2 (2 - \theta) - e^{-2\lambda} (1 + \lambda)^2 - \theta e^{-\lambda} (2 + 2\lambda + \lambda^2 - 2\theta(1 + \lambda))))^{1/2},$$

$$sp_2(\theta, \lambda) = 2\theta^2 e^{3\lambda} (3 - 3\theta + \theta^2) - 2(1 + \lambda)^3 - 3\theta e^{\lambda} (1 + \lambda)(2 + 2\lambda + \lambda^2 - 2\theta(1 + \lambda)),$$

$$sp_3(\theta, \lambda) = \theta e^{2\lambda} (6 + 6\lambda + 3\lambda^2 + \lambda^3 + 6\theta^2 (1 + \lambda) - 3\theta(2 + \lambda)^2),$$

$$kp_1(\theta, \lambda) = -\theta^2 e^{4\lambda} (24 + 24\lambda + 12\lambda^2 + 4\lambda^3 + \lambda^4),$$

$$kp_2(\theta, \lambda) = 3\theta^2 e^{5\lambda} (-8 + 8\theta - 4\theta^2 + \theta^3) + 3e^{3\lambda} (1 + \lambda)^4,$$

$$kp_3(\theta, \lambda) = 6\theta e^{2\lambda} (1 + \lambda)^2 (2 + 2\lambda + \lambda^2 - 2\theta(1 + \lambda)).$$

$$kp_4(\theta, \lambda) = 2\theta e^{3\lambda} (1 + \lambda)(9\theta^2 (1 + \lambda) - 6\theta(3 + 3\lambda + \lambda^2) + 2(6 + 6\lambda + 3\lambda^2 + \lambda^3)),$$

$$kp_5(\theta, \lambda) = -2\theta^2 e^{4\lambda} (6\theta^2 (1 + \lambda) - 3\theta(6 + 6\lambda + \lambda^2) + 2(12 + 12\lambda + 3\lambda^2 + \lambda^3)).$$
The shape of the exponential distribution with stabilizer constant has shown in Figure 1 for probability distribution function and Figure 2 for cumulative distribution function which various parameters $\lambda$. This new distribution with support in $(0,1]$ has no shape parameter since its only has one shape, this can be seen throughout the formula of skewness and kurtosis depend only on $\lambda$. The graph of this probability distribution function is always convex as it is stretched to the right axis to $x = 1$ as $\lambda$ decreases in value with scale parameter $1/\lambda$.

![Figure 1: Probability density function of random variables $W$ which various parameters $\lambda$.](image1)

![Figure 2: Cumulative distribution function of random variables $W$ which various parameters $\lambda$.](image2)

### 3 Probability Density Function for Convolution of Generated Random Variable from Exponential Distribution

The aim of this section is to present probability density function and its properties for convolution of generating random variables from an exponential distribution with stabilizer constant.

**Theorem 3.1** Let $W_1, W_2, ..., W_n$ be $n$ independent and identically exponential distributions with stabilizer constant where the probability density function is defined as follows

$$f(w_i; \lambda) = \theta \lambda e^{-\lambda w_i}$$

for positive $\lambda$, $0 < w_i \leq 1$ and $\theta = 1/(1-e^{-\lambda})$. Then the sum of random variables $S_n = W_1 + W_2 + ... + W_n$ has probability density function

$$f(s_n; \lambda) = \frac{\theta^n \lambda^n}{(n-1)!} s_n^{n-1} e^{-\lambda s_n} \text{ for } 0 < s_n \leq n.$$  

**Proof.** This theorem is proved by using analytical direct methods and properties of expectation. We start by using definition of joint distribution of random variables $W_1, W_2, ..., W_n$ from independent and identically exponential distribution with stabilizer constant, that is...
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\[ f(w_1, w_2, ..., w_n) = \theta^n \lambda^n e^{-\lambda(w_1^2 + w_2^2 + ... + w_n^2)}. \]

Let \( h \) is continuous function on the real line and we define an identity as follows

\[ I_n(h) = E(h(w_1 + w_2 + ... + w_n)). \]

Let \( w_i = y_i^2 \) for \( 0 < y_i \leq 1 \) and \( i = 1, 2, ..., n \) then it holds

\[
I_n(h) = \theta^n \lambda^n \int_0^1 \cdots \int_0^1 h(w_1 + w_2 + ... + w_n) e^{-\lambda(w_1^2 + w_2^2 + ... + w_n^2)} \, dw_1 \, dw_2 \cdots dw_n
\]

\[
= \theta^n \lambda^n 2^n \int_0^1 \cdots \int_0^1 g(y) (y_1 y_2 \cdots y_n) e^{-\lambda(y_1^2 + y_2^2 + ... + y_n^2)} \, dy_1 \, dy_2 \cdots dy_n
\]

where it is defined function \( g(y) = h(y_1^2 + y_2^2 + ... + y_n^2) \) to simplify notation.

Now we might change \( n \) axes of cartesian coordinates \((y_1, y_2, y_3, ..., y_{n-1}, y_n)\) to spherical coordinates by following form

\[
y_1 = r \sin \omega_{n-1} \sin \omega_2 \sin \omega_1,
\]

\[
y_2 = r \sin \omega_{n-1} \sin \omega_2 \cos \omega_1,
\]

\[
y_3 = r \sin \omega_{n-1} \cos \omega_2,
\]

\[
..., \n\]

\[
y_{n-1} = r \sin \omega_{n-1} \cos \omega_{n-2},
\]

\[
y_n = r \cos \omega_{n-1}
\]

where \( r \) as spherical radius that is selected for \( 0 < r \leq n \), \( 0 < \omega_k \leq \pi/2 \) for \( k = 1, 2, ..., n-1 \). Furthermore, let us define \( r_k^2 = y_1^2 + y_2^2 + ... + y_k^2 \) and \( r_k = r \) such that it is obtained \( \cos \omega_k = y_{k+1}/r_{k+1} \) and \( \sin \omega_k = r_k/r_{k+1} \) for \( 0 < r_k \leq k \).

Then we have the Jacobian from changing variable process as follows

\[
|J| = r^{n-1} \sin^{n-2} \omega_{n-1} \sin^{n-3} \omega_{n-2} \cdots \sin \omega_2
\]

\[
= r^{n-1} \prod_{k=1}^{n-1} \sin^{k-1} \omega_k.
\]

and it is also obtained

\[
\prod_{k=1}^{n-1} y_j = r^n \prod_{k=1}^{n-1} \sin^{k-1} \omega_k \cos \omega_k.
\]

We have defined \( S_n \) as convolution of random variables \( W_i \) in term of \( S_n = \sum_{i=1}^n W_i \) where \( 0 < w_i \leq 1 \) then support of random variables \( S_n \) is \( (0, n] \).

Since we have \( r^2 = S_n \), then we can rewrite the identity \( I_n(h) \) with boundary integral for \( r \) in \( (0, n] \) as follows
The process of changing variables \( r^2 = S_n \) leads us to write identity \( I_n(h) \) in the following form

\[
I_n(h) = \frac{\theta^n \lambda^n}{(n-1)!} \int_0^{S_n} h(r^2) r^{2n-2} e^{-\lambda r^2} dr^2
\]

This identity gives the expectation of random variable \( S_n \) with probability density function

\[
f(s_n; \lambda) = \frac{\theta^n \lambda^n}{(n-1)!} s_n^{n-1} e^{-\lambda s_n}, \quad \text{for } 0 < s_n \leq n.
\]

**Corollary 3.2** Let \( W_1, W_2, ..., W_n \) be \( n \) independent and identically exponential distributions with stabilizer constant where the probability density function is defined as follows

\[
f(w_i; \lambda) = \theta \lambda e^{-\lambda w_i}
\]

for positive \( \lambda, \ 0 < w_i \leq 1 \) and \( \theta = 1/(1-e^{-\lambda}) \). Then the sum of random variables \( S_n = W_1 + W_2 + ... + W_n \) for \( 0 < s_n \leq n \) with probability density function

\[
f(s_n; \lambda) = \frac{\theta^n \lambda^n}{(n-1)!} s_n^{n-1} e^{-\lambda s_n}
\]

has cumulative distribution function

\[
F(s_n; \lambda) = \frac{\theta^n}{(n-1)!} \Gamma(n, \lambda s_n).
\]

**Proof.** The cumulative distribution function is derived for \( 0 < t \leq s_n \), then we have

\[
F(s_n; \lambda) = \int_0^{s_n} \frac{\theta^n \lambda^n}{(n-1)!} t^{n-1} e^{-\lambda t} dt.
\]

Let we use change variable \( y = \lambda t \), then we obtain
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\[ F(s_n; \lambda) = \frac{\theta^n}{(n-1)!} \int_0^{\frac{2s_n}{\lambda}} y^{n-1} e^{-y} dy \]
\[ = \frac{\theta^n}{(n-1)!} \Gamma(n, \lambda s_n) \]

where \( \Gamma(n, \lambda S_n) \) is the lower incomplete gamma function. ■

The properties of convolution distribution from generated random variable from an exponential distribution with stabilizer constant are given in expression of mean, variance, moment generating function and characteristic function respectively as follows

\[ \mu_{s_n} = \frac{\theta^n}{\lambda(n-1)!} \Gamma(n+1, \lambda n), \]
\[ \sigma_{s_n}^2 = \frac{\theta^n}{\lambda^2 (n-1)!} \left( \Gamma(n+2, \lambda n) - \frac{\theta^n}{\lambda(n-1)!} \Gamma(n+1, \lambda n)^2 \right), \]
\[ M_{s_n}(t) = \frac{\theta^n \lambda^n}{(n-1)! (\lambda - t)^n} \Gamma(n,n(\lambda - t)) \text{ for } -\infty < t < \infty, \]
\[ \phi_{s_n}(t) = \frac{\theta^n \lambda^n}{(n-1)! (\lambda - it)^n} \Gamma(n,n(\lambda - it)) \text{ for } -\infty < t < \infty, \]

The most informative properties to express the shape of the convolution distribution are skewness and kurtosis respectively in the following form

\[ \text{Skew}(S_n) = \frac{sc_1(\theta, \lambda) + sc_2(\theta, \lambda) + sc_3(\theta, \lambda)}{(-\theta^n)^{3/2} \left( \theta^n \Gamma(n+1, \lambda n)^2 - (n-1)! \Gamma(n+2, \lambda n) \right)^{3/2}}, \]
\[ \text{Kurt}(S_n) = \frac{kc_1(\theta, \lambda) + kc_2(\theta, \lambda) + kc_3(\theta, \lambda)}{(\theta^n \Gamma(n+1, \lambda n)^2 - (n-1)! \Gamma(n+2, \lambda n))^3}, \]

where functions in the form of skewness and kurtosis are defined as follows

\[ sc_1(\theta, \lambda) = 2\theta^n \Gamma(n+1, \lambda n)^3, \]
\[ sc_2(\theta, \lambda) = -3\theta^{2n} (n-1)! \Gamma(n+1, \lambda n) \Gamma(n+2, \lambda n), \]
\[ sc_3(\theta, \lambda) = \theta^n (n-1)!^2 \Gamma(n+3, \lambda n), \]
\[ kc_1(\theta, \lambda) = -3\theta^{2n} \Gamma(n+1, \lambda n)^4, \]
\[ kc_2(\theta, \lambda) = 6\theta^n (n-1)! \Gamma(n+1, \lambda n)^2 \Gamma(n+2, \lambda n), \]
\[ kc_3(\theta, \lambda) = -4(n-1)!^2 \Gamma(n+1, \lambda n) \Gamma(n+3, \lambda n), \]
\[ kc_4(\theta, \lambda) = \theta^n (n-1)!^3 \Gamma(n+4, \lambda n). \]
The shapes of convolution distribution of generating random variables from an exponential distribution with stabilizer constant have shown in Figure 3 for the probability density function which various parameters $\lambda$ and Figure 5 which various $n$-fold convolution, Figure 4 for cumulative distribution function which various parameters $\lambda$ and Figure 6 which various $n$-fold convolution. This distribution with support in $(0,n]$ with $n$ as number of convolutions has shape parameter $n$ and scale parameter $1/\lambda$. This shape and scale parameters can be seen throughout the formula of skewness and kurtosis depend on $n$ and $\lambda$.

Figure 3: Probability density function of random variables $S_n$ which various parameters $\lambda$.

Figure 4: Cumulative distribution of random variables $S_n$ which various parameter $\lambda$.

Figure 5: Probability density function of random variables $S_n$ which various $n$-fold convolution.

Figure 6: Cumulative distribution of random variables $S_n$ which various $n$-fold convolution.

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