Tree Cover of the Join and the Corona of Graphs

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Abstract

Let $G$ be a graph and $T_G = \{G_1, G_2, G_3, \ldots, G_k\}$ be a collection of subgraphs of $G$ where $G_i$ is a subtree of $G$ for every $i \in \{1, 2, \ldots, k\}$. If for every edge $e \in E(G)$, there exists $G_i \in T_G$ such that $e \in E(G_i)$, then $T_G$ is a tree cover of $G$. The tree covering number of $G$ is the minimum cardinality among the tree covers of $G$. In this paper, we establish some bounds for the tree covering numbers of the join and the corona of two vertex-disjoint graphs.

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1 Introduction

Let $G$ be a simple graph. A subtree of $G$ is a connected acyclic subgraph of $G$. A collection $T_G = \{G_1, G_2, G_3, \ldots, G_k\}$ of subgraphs of $G$ is a tree cover of $G$ if $G_i$ is a subtree of $G$ for every $i \in \{1, 2, \ldots, k\}$ and for every edge $e \in E(G)$, there exists $G_i \in T_G$ such that $e \in E(G_i)$. The tree covering number of $G$, denoted by $t_c(G)$, is given by $t_c(G) = \min\{|T_G|: T_G \text{ is a tree cover of } G\}$.

The graph $G$ in Figure 1.1 has tree covering number equal to 2.
Indeed, let $\mathcal{F}_G = \{G_1, G_2\}$, where $G_1$ and $G_2$ are subgraphs of $G$ shown in Figure 1.2 and Figure 1.3, respectively.

Figure 1.1: A graph $G$ with $t_c(G) = 2$.

Figure 1.2: The graph $G_1$.

Figure 1.3: The graph $G_2$. 
Clearly, $G_1$ and $G_2$ are subtrees of $G$. Moreover, $G_1 \cup G_2 = G$. Hence, every edge of $G$ is either in $G_1$ or in $G_2$. Consequently, $\mathcal{S}_G$ is a tree cover of $G$. Thus, by definition, $t_e(G) \leq |\mathcal{S}_G| = 2$. Since $G$ is not a tree, Theorem 2.2 found in [1] asserts that $t_e(G) \geq 2$. Combining the two inequalities gives $t_e(G) = 2$.

The next section establishes an upper bound for the tree covering number of the join of two vertex-disjoint graphs.

2 Tree Covering Number of the Join of Graphs

Here, we formally define the join of two vertex-disjoint graphs.

Definition 2.1 [4] Let $G$ and $H$ be vertex-disjoint graphs. The join $G \oplus H$ of $G$ and $H$ has vertex-set $V(G \oplus H) = V(G) \cup V(H)$ and edge-set

$$E(G \oplus H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$$

Consequently,

$$|V(G \oplus H)| = |V(G)| + |V(H)|,$$

and

$$|E(G \oplus H)| = |E(G)| + |E(H)| + |V(G)||V(H)|.$$

Note also that the operation $\oplus$ is commutative, i.e., $G \oplus H$ is isomorphic to $H \oplus G$ with respect to adjacency.

Let us consider an illustration of the above definition.

Example 2.2 Consider the path $P_3$ and the complete graph $K_1$ with vertex-set $V(K_1) = \{u\}$. Then the join of $P_3$ and $K_1$ is shown below.

![Figure 2.1: The join $P_3 + K_1$.](image)

In the above illustration, every vertex of $P_3$ is joined with the vertex $u$.

An upper bound for the tree covering number of the join of two vertex-disjoint graphs is established in the following theorem.
Theorem 2.3 Let $G$ and $H$ be two vertex-disjoint graphs of orders $m$ and $n$, respectively. Then,

$$t_c(G \oplus H) \leq \min \{m + t_c(H), n + t_c(G)\}.$$

Proof: Let $V(G) = \{v_1, v_2, v_3, \ldots, v_m\}$ and $V(H) = \{u_1, u_2, u_3, \ldots, u_n\}$. Then $v_i u_j \in E(G \oplus H)$ for all $i = 1, 2, 3, \ldots, m$ and for all $j = 1, 2, 3, \ldots, n$. Let $\mathcal{T}_G$ and $\mathcal{T}_H$ be tree covers of $G$ and $H$, respectively, such that $|\mathcal{T}_G| = t_c(G)$ and $|\mathcal{T}_H| = t_c(H)$. Now, for every vertex $v \in V(G)$, $\{v\} \oplus H \cup \overline{H}$ is a star, and hence a tree. Moreover, there exists $G_v \in \mathcal{T}_G$ with $v \in V(G)$ and $G_v \cup \{v\} \oplus H \cup \overline{H}$ is a tree. Let

$$\mathcal{T}_{G+H} = \mathcal{T}_H \cup \{G_v \cup \{v\} \oplus H \cup \overline{H} : v \in V(G)\}.$$ 

Then $\mathcal{T}_{G\oplus H}$ is a tree cover of $G \oplus H$. Thus,

$$t_c(G \oplus H) \leq |\mathcal{T}_H| + |\{G_v \cup \{v\} \oplus H \cup \overline{H} : v \in V(G)\}|.$$ 

But $|\mathcal{T}_H| = t_c(H)$ and $|\{G_v \cup \{v\} \oplus H \cup \overline{H} : v \in V(G)\}| = |V(G)| = m$. Thus,

$$t_c(G \oplus H) \leq m + t_c(H). \quad (1)$$

Similarly,

$$t_c(G \oplus H) \leq n + t_c(G). \quad (2)$$

Combining Inequalities (1) and (2) gives the desired result. \(\square\)

Consider now the complete bipartite graph $K_{m,n}$. An upper bound for its tree covering number is established in the following theorem.

Theorem 2.4 Let $m$ and $n$ be positive integers. Then, $t_c(K_{m,n}) \leq \min \{m, n\}$.

Proof: Note that $K_{m,n} = \overline{K_m} \oplus \overline{K_n}$. Thus by Theorem 2.3,

$$t_c(K_{m,n}) = t_c(\overline{K_m} \oplus \overline{K_n}) \leq \min \{m + t_c(\overline{K_n}), n + t_c(\overline{K_m})\}$$

$$= \min \{m + 0, n + 0\}$$

$$= \min \{m, n\}.$$ 

Thus, $t_c(K_{m,n}) \leq \min \{m, n\}$. \(\square\)

The tree covering number of the corona of graphs is established in the following section.
3 Tree Covering Number of the Corona of Graphs

Formally, we define the corona of two vertex-disjoint graphs.

Definition 3.1 [4] The corona $G \circ H$ of two graphs $G$ and $H$ is the graph obtained by taking one copy of $G$ of order $n$ and $n$ copies of $H$, and then joining the $i^{th}$ vertex of $G$ to every vertex in the $i^{th}$ copy of $H$.

Example 3.2 The figure below illustrates the corona $P_3 \circ C_3$.

![Figure 3.1: The corona $P_3 \circ C_3$.](image)

Note that if $H$ is a tree, then $G \circ H$ has tree covering number equal to the tree covering number of $G$. We formally write this result in the following theorem.

Theorem 3.3 Let $G$ be a connected graph. Then for any tree $T$, $t_c(G \circ T) = t_c(G)$.

Next, we give an upper bound for the tree covering number of the corona of two graphs as a linear combination of the tree covering number of $G$ and the tree covering number of $H$.

Theorem 3.4 Let $G$ and $H$ be two nontrivial connected graphs of orders $m$ and $n$, respectively. Then $t_c(G \circ H) \leq t_c(G) + mt_c(H)$.

Proof: Let $\mathcal{F}_G$ and $\mathcal{F}_H$ be tree covers of $G$ and $H$, respectively, such that $|\mathcal{F}_G| = t_c(G)$ and $|\mathcal{F}_H| = t_c(H)$. For every $u \in V(G)$, each copy $H_u$ of $H$ can be covered by $|\mathcal{F}_H|$ subtrees. Hence, all the copies of $H$ in $G \circ H$ can be covered by $mt_c(H)$ subtrees. Now, for every vertex $v \in V(G)$, there exists $G_v \in \mathcal{F}_G$ with $v \in G_v$ and $G_v \cup \{v\} \oplus H \cup \overline{H}$ is a tree. Thus, the family $\mathcal{F}_{G\circ H} = \bigcup\{\mathcal{F}_{H_u} : u \in V(G)\} \cup \{G_v \cup \{v\} \oplus H \cup \overline{H}\}$ is a tree cover of $G \circ H$. Accordingly,

$$t_c(G \circ H) \leq \left| \bigcup\{\mathcal{F}_{H_u} : u \in V(G)\} \cup \{G_v \cup \{v\} \oplus H \cup \overline{H}\} \right|$$

$$= m|\mathcal{F}_H| + |\mathcal{F}_G|$$

$$= mt_c(H) + t_c(G).$$

This completes the proof. $\square$
References


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