Resonance Problems for the Weighted \( p \)-Biharmonic Operator

K. Ben Haddouch, N. Tsouli and El Miloud Hssini

University Mohamed I, Faculty of Sciences
Department of Mathematics and Computer Sci., Oujda, Morocco

Z. El Allali

University Mohamed I, Faculty Multidisciplinary of Nador,
Department of Mathematics and Computer Sci., Morocco

Copyright © 2014 K. Ben Haddouch, N. Tsouli, El Miloud Hssini and Z. El Allali. This article is distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

In this work, by using the Mountain-Pass theorem with the Cerami Palais-Smale condition, we are interested at the existence of nontrivial solutions of the problem (1) governed by the weighted \( p \)-biharmonic operator.

Mathematics Subject Classification: 35A15, 35J40

Keywords: \( p \)-biharmonic operator, resonance, Cerami (PS) condition, Mountain-Pass theorem

1 Introduction and main results

In the present paper, we are concerned with the existence of weak solutions of the following problem

\[
\begin{aligned}
\Delta(p|\Delta u|^{p-2}\Delta u) &= \lambda_1 m(x)|u|^{p-2}u + f(x, u) \quad \text{in} \quad \Omega \\
u &= \Delta u = 0 \quad \text{on} \quad \partial\Omega,
\end{aligned}
\]  

(1)
where $p > 1$, $\Omega$ is a bounded domain of $\mathbb{R}^N$ ($N \geq 1$) with smooth boundary $\partial \Omega$, $\rho \in C(\overline{\Omega})$ with $\inf \Omega \rho(x) > 0$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a bounded Carathéodory function, $m \in C(\overline{\Omega})$ is a positive weight function and $\lambda_1$ denote the first eigenvalue for the following eigenvalue problem

$$\begin{cases}
\Delta(\rho|\Delta u|^{p-2}\Delta u) = \lambda m(x)|u|^{p-2}u & \text{in } \Omega \\
u = \Delta u = 0 & \text{on } \partial \Omega.
\end{cases} \quad (2)$$

The investigation of existence of solutions for problems at resonance has drawn the attention of many authors, see for example [3, 4, 1, 8, 12, 13, 16] and references therein. Recently the authors in [10] interested at the fourth order problems governed by the weighted p-biharmonic operator

$$\begin{cases}
\Delta(\rho(x)|\Delta u|^{p-2}\Delta u) = \lambda_1 m(x)|u|^{p-2}u + f(x, u) - h(x) & \text{dans } \Omega \\
u = \Delta u = 0 & \text{sur } \partial \Omega.
\end{cases} \quad (3)$$

Using the Landesman-Lazer conditions, the authors proved the existence of at least a nontrivial solutions of problem (3). In [11], Liu and Squassina have studied the following p-biharmonic problem

$$\begin{cases}
\Delta(\Delta u|^{p-2}\Delta u) = g(x, u) & \text{in } \Omega \\
u = \Delta u = 0 & \text{on } \partial \Omega.
\end{cases} \quad (4)$$

Under some conditions on $g(x, u)$ at resonance, the authors established the existence of at least one nontrivial solution.

According to the work of M. Talbi and N. Tsouli [15], the eigenvalue problem (2) has a nondecreasing and unbounded sequence of eigenvalues, and the first eigenvalue $\lambda_1$ characterized as

$$\lambda_1 = \inf \left\{ \int_\Omega \rho|\Delta u|^p dx : \int_\Omega m(x)|u|^p dx = 1 \right\},$$

where $X := W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, the reflexive Banach space endowed with the norm

$$||u|| = \left( \int_\Omega \rho|\Delta u|^p dx \right)^{1/p}.$$ 

When $m \in C(\overline{\Omega})$ and $m \geq 0$, the authors showed that $\lambda_1$ is positive, simple and isolated. Therefore

$$\int_\Omega \rho|\Delta u|^p dx \geq \lambda_1 \int_\Omega m(x)|u|^p dx \quad \text{for all } u \in X. \quad (4)$$

Moreover, there exists a unique positive eigenfunction $\varphi_1$ which can be chosen normalized.

We may now assume the following conditions:
Resonance problems for the weighted $p$-biharmonic operator

$(H_1)$ we have

$$\limsup_{t \to 0} \frac{pF(x,t)}{m(x)|t|^p} \leq -\lambda_1$$ uniformly for all $x \in \Omega$,

where $F(x,t) = \int_0^t f(x,s)ds$, for almost every $x \in \Omega$, $\forall t \in \mathbb{R}$,

$(H_2)$ there exists a function $h : \mathbb{R}^+ \to \mathbb{R}^+$ with

$$\liminf \frac{h(a_nb_n)}{h(b_n)} > 0, \ h(b_n) \to \infty \text{ when } a_n \to a > 0 \text{ and } b_n \to +\infty, \ (5)$$

and another function $\mu(x) \in L^\infty(\Omega)$ such that

$$\liminf_{||t|| \to \infty} \frac{pF(x,t) - f(x,t)t}{h(||t||)} \geq \mu(x) > 0, \text{ for almost every } x \in \Omega. \ (6)$$

We have the main result

**Theorem 1.1** Assume that $(H_1)$ and $(H_2)$ hold. Then problem (1) has at least a weak nontrivial solution.

2 Preliminaries and proof of theorems

We consider the following energy functional $\Phi : X \to \mathbb{R}$ defined by

$$\Phi(u) = \frac{1}{p} \int_\Omega \rho|\Delta u|^p dx - \frac{\lambda_1}{p} \int_\Omega m(x)|u|^p dx - \int_\Omega F(x,u) dx.$$

It is well known that $\Phi \in \mathcal{C}^1(X,\mathbb{R})$, with derivative at point $u \in X$ is given by

$$\langle \Phi'(u), v \rangle = \int_\Omega \rho|\Delta u|^{p-2} \Delta u \Delta v dx - \lambda_1 \int_\Omega m(x)|u|^{p-2} uv dx - \int_\Omega f(x,u)v dx,$$

for every $v \in X$, and its critical points are in fact weak solutions to problem (1).

In order to use variational methods, we give some results related to the Cerami Palais-Smale condition. Recall that a functional $\Phi$ satisfies the Cerami Palais-Smale condition on $X$, if for any sequence $(u_n)$ such that

$$|\Phi(u_n)| \leq c \text{ and } (1 + ||u_n||) \langle \Phi'(u_n), \varphi \rangle \to 0 \text{ for all } \varphi \in X,$$

we can show that there exists a convergent subsequence.

In the sequel, we show that the functional $\Phi$ has the mountain pass geometry. This purpose is proved in the next lemma.
**Lemma 2.1** Suppose that \((H_1)\) and \((H_2)\) hold, then we have

(i) There exist \(r, \rho > 0\) such that \(\inf_{||u||=r} \Phi(u) \geq \rho > 0\).

(ii) There exists a nonnegative function \(e \in X\) such that \(||e|| > r\) and \(\Phi(e) < 0\).

**Proof 2.2** (i) From \((H_1)\) and the boundedness of \(f\), for every \(\varepsilon > 0\), we have

\[
pF(x, u) \leq (-\lambda_1 + \varepsilon)m(x)|u|^p + pM|u|,
\]
uniformly for all \(x \in \Omega, \forall u \in \mathbb{R}\), where \(M\) is a nonnegative constant. Then we obtain

\[
\Phi(u) \geq \frac{1}{p} \int_{\Omega} \rho|\Delta u|^p dx - \frac{\lambda_1}{p} \int_{\Omega} m(x)|u|^p dx - \frac{(-\lambda_1 + \varepsilon)}{p} \int_{\Omega} m(x)|u|^p dx - M \int_{\Omega} |u| dx \geq \frac{1}{p} \left(1 - \varepsilon c_p^p \|m\|_{\infty}\right) \|u\|^p - c_1 M \|u\|
\]

where \(c_p\) and \(c_1\) denote the embedding constants of Sobolev \(X \hookrightarrow L^p(\Omega), X \hookrightarrow L^1(\Omega)\) respectively. We take \(\varepsilon < \frac{1}{c_p^p \|m\|_{\infty}}\) and choosing \(||u|| = r\) small enough, we can obtain a positive constant \(\rho\) such that \(\Phi(u) \geq \rho\) as \(||u|| = r\).

(ii) We will show that there exists some \(t \in \mathbb{R}\) such that \(\Phi(t\varphi_1) < 0\). Suppose, by contradiction that there exists a sequence \(t_n \in \mathbb{R}\) such that \(t_n \to \infty\) as \(n \to \infty\) and \(\Phi(t_n\varphi_1) \geq 0\). For \(u \in \mathbb{R}\) large enough, one has

\[
\frac{\partial}{\partial u} \left(-\frac{F(x, u)}{u^p}\right) = \frac{pF(x, u) - f(x, u)u}{u^{p+1}} - \frac{pF(x, u) - f(x, u)h(|u|)}{h(|u|)} \frac{1}{u^{p+1}} \geq (\mu(x) - \varepsilon) \frac{1}{u^{p+1}} = \frac{(\mu(x) - \varepsilon)}{p} \frac{\partial}{\partial u} \left(-\frac{1}{u^p}\right).
\]

Then

\[
\int_t^s \frac{\partial}{\partial u} \left(-\frac{F(x, u)}{u^p}\right) du \geq \int_t^s \frac{(\mu(x) - \varepsilon)}{p} \frac{\partial}{\partial u} \left(-\frac{1}{u^p}\right) du.
\]

Since, the function \(f\) is bounded, take \(s \to \infty\), for \(t \in \mathbb{R}\) large enough, we obtain

\[
F(x, t) \geq \frac{\mu(x)}{p}.
\]
Using the definition of \( \varphi_1 \) and the fact that

\[
\limsup_{n \to \infty} \Phi(t_n \varphi_1) \geq \liminf_{n \to \infty} \Phi(t_n \varphi_1) \geq 0,
\]

we get

\[
\limsup_{t_n \to \infty} \int_{\Omega} -F(x, t_n \varphi_1(x)) \, dx \geq 0,
\]

and it follows from (7) that

\[
\int_{\Omega} \mu(x) \, dx \leq 0,
\]

which is a contradiction with (6) and the assertion (ii) is proved by choosing \( e = t \varphi_1 \) for some \( t \in \mathbb{R} \).

**Lemma 2.3** Assume that \((H_1)\) and \((H_2)\) hold. Then the functional \( \Phi \) satisfies the Cerami Palais-Smale condition on \( X \).

**Proof 2.4** Let \((u_n)\) be a sequence in \( X \), and \( c \) a real number such that:

\[
|\Phi(u_n)| \leq c, \tag{8}
\]

and

\[
\langle \Phi(u_n), \varphi \rangle \leq \varepsilon_n \frac{||\varphi||}{1 + ||u_n||}. \tag{9}
\]

We claim that \((u_n)\) is bounded in \( X \). Indeed, we proceed by contradiction, suppose that

\[
||u_n|| \to +\infty, \text{ as } n \to +\infty.
\]

Put \( v_n = u_n/||u_n|| \), thus \((v_n)\) is bounded, for a subsequence still denoted \((v_n)\), we can assume that \( v_n \to v \) weakly in \( X \), by the Sobolev imbedding theorem we have \( v_n \to v \) strongly in \( L^p(\Omega) \). Dividing (8) by \( ||u_n||^p \), we get

\[
\lim_{n \to +\infty} \left( \frac{1}{p} \int_{\Omega} \rho|\Delta v_n|^p \, dx - \frac{\lambda_1}{p} \int_{\Omega} m(x)|v_n|^p \, dx - \int_{\Omega} \frac{F(x, u_n)}{||u_n||^p} \, dx \right) = 0. \tag{10}
\]

By the hypotheses on \( f \), we obtain

\[
\lim_{n \to +\infty} \int_{\Omega} \frac{F(x, u_n)}{||u_n||^p} \, dx = 0,
\]

while

\[
\lim_{n \to +\infty} \int_{\Omega} m(x)|v_n|^p \, dx = \int_{\Omega} m(x)|v|^p \, dx,
\]

from (10), we deduce that

\[
\lim_{n \to +\infty} \int_{\Omega} \rho|\Delta v_n|^p \, dx = \lambda_1 \int_{\Omega} m(x)|v|^p \, dx.
\]
Which together with the variational characterization of $\lambda_1$ and the weak lower semi continuity of norm yield

$$\lambda_1 \int_{\Omega} m(x)|v|^p dx \leq \int_{\Omega} \rho|\Delta v|^p dx \leq \liminf_{n \to +\infty} \int_{\Omega} \rho|\Delta v_n|^p dx = \lambda_1 \int_{\Omega} m(x)|v|^p dx,$$

which implies that

$$v_n \to v \text{ strongly in } X, \quad \text{and} \quad \int_{\Omega} \rho|\Delta v|^p dx = \lambda_1 \int_{\Omega} m(x)|v|^p dx.$$

On the other hand, by (8) we have

$$-pc \leq \int_{\Omega} \rho|\Delta u_n|^p dx - \lambda_1 \int_{\Omega} m(x)|u_n|^p dx - p \int_{\Omega} F(x, u_n) dx \leq pc. \quad (11)$$

Moreover, in view of (9), one can also have

$$-\varepsilon_n \frac{||u_n||}{1 + ||u_n||} \leq - \int_{\Omega} \rho|\Delta u_n|^p dx + \lambda_1 \int_{\Omega} m|u_n|^p dx + \int_{\Omega} f(x, u_n) u_n dx \leq \varepsilon_n \frac{||u_n||}{1 + ||u_n||}. \quad (12)$$

Summing up and dividing by $h(||u_n||)$, we obtain

$$-pc - \varepsilon_n \frac{||u_n||}{1 + ||u_n||} \leq \int_{\Omega} \frac{pF(x, u_n) dx - f(x, u_n) u_n h(||v_n|| ||u_n||)}{h(||u_n||) h(||u_n||)} dx \leq \frac{pc + \varepsilon_n ||u_n||}{h(||u_n||)}. \quad (13)$$

From this we can see that

$$\liminf_{n \to +\infty} \int_{\Omega} \frac{pF(x, u_n) dx - f(x, u_n) u_n h(||v_n|| ||u_n||)}{h(||u_n||) h(||u_n||)} dx \leq 0.$$

Using Fatou’s lemma, we obtain a contradiction with $(H_2)$. Thus, we conclude that $(u_n)$ is bounded in $X$. For a subsequence denoted also $(u_n)$, there exists $u \in X$ such that $u_n \rightharpoonup u$ weakly in $X$ and strongly in $L^p(\Omega)$. Since

$$\lim_{n \to +\infty} \langle \Phi'(u_n), (u_n - u) \rangle = 0,$$

we have

$$\langle \Phi'(u_n), (u_n - u) \rangle = \int_{\Omega} \rho|\Delta u_n|^{p-2} \Delta u_n \Delta (u_n - u) dx - \lambda_1 \int_{\Omega} |u_n|^{p-2} u_n (u_n - u) dx$$

$$- \int_{\Omega} f(x, u_n) (u_n - u) dx = o_n(1)$$

Using the hypotheses on $m$ and $f$, we see that

$$\lim_{n \to +\infty} \int_{\Omega} m(x)|u_n|^{p-2} u_n (u_n - u) dx = 0, \quad \lim_{n \to +\infty} \int_{\Omega} f(x, u_n) (u_n - u) dx = 0.$$
Consequently,
\[
\lim_{n \to +\infty} \int_{\Omega} \rho |\Delta u_n|^{p-2} \Delta u_n \Delta (u_n - u) \, dx = 0.
\]
In the same way, we obtain
\[
\lim_{n \to \infty} \int_{\Omega} \rho |\Delta u|^{p-2} \Delta u \Delta (u_n - u) \, dx = 0.
\]
Therefore, the Hölder inequality imply that
\[
0 = \lim_{n \to \infty} \int_{\Omega} (\rho |\Delta u_n|^{p-2} \Delta u_n - \rho |\Delta u|^{p-2} \Delta u) \Delta (u_n - u) \, dx
\geq \lim_{n \to \infty} \left[ ||u_n||^p - \left( \int_{\Omega} \rho |\Delta u_n|^p \, dx \right)^{p-1} \left( \int_{\Omega} \rho |\Delta u|^p \, dx \right)^{1/p} \right.
\left. - \left( \int_{\Omega} \rho |\Delta u|^p \, dx \right)^{p-1} \left( \int_{\Omega} \rho |\Delta u_n|^p \, dx \right)^{1/p} \right] + ||u||^p
= \lim_{n \to \infty} (||u_n||^{p-1} - ||u||^{p-1})(||u_n|| - ||u||) \geq 0,
\]
hence $||u_n|| \to ||u||$. By the uniform convexity of $X$, it follows that $u_n \to u$ strongly in $X$ and $I$ satisfies the Cerami Palais-Smale condition.

Proof 2.5 (Proof of Theorem 1.1) From Lemmas 2.1 and 2.3, it is clear to see that $\Phi$ satisfies the hypotheses of Mountain-Pass theorem. Therefore $\Phi$ admits a critical value.

References


Received: August 17, 2014; Published: June 21, 2015