Common Coupled Coincidence Point for Coincidentally Commuting and Property (E.A.) Maps in IFMS without Completeness

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Abstract

In this paper, we define property(E.A.) with the counterpart of the notion of property(E.A.) in Park[13], and prove a common coupled coincidence point theorem for two pairs of coincidentally commuting and property(E.A.) maps in IFMS without completeness.

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1. INTRODUCTION

In 2006, Bhaskar et.al.[1] proved to coupled contraction mapping theorem, and this result was generalized to coupled coincidence point theories([3],[9]) under two separate sets of conditions. Zhu et.al[16] studied coupled fixed point theorems in fuzzy metric spaces in which they obtained a fuzzy version of the result of [1]. Also, Hu[8] obtained common coupled fixed point results in fuzzy metric spaces. Furthermore, Choudhury et.al.[2] proved a coupled coincidence point theorem for coincidentally commuting maps in fuzzy metric spaces. Thus coupled fixed point problems have been studied in structures which are generalization of metric spaces, in probabilistic metric spaces, in G-metric spaces and fuzzy metric spaces.
Park et. al. [11] defined an IFMS and proved a fixed point theorem in IFMS, Park [12] studied a fixed point for common property (E.A.) and weak compatible maps, and obtained some common fixed point for the weakly commuting maps. In this paper, we define property (E.A.) with the counterpart of the notion of property (E.A.) in Park [13], and prove a common coupled coincidence point theorem for two pairs of coincidentally commuting and property (E.A.) maps in IFMS without completeness.

2. Preliminaries

In this section, we recall some definitions, properties and known results in the intuitionistic fuzzy symmetric space as following:

Let us recall (see [15]) that a continuous $t-$norm is a operation $\ast : [0, 1] \times [0, 1] \rightarrow [0, 1]$ which satisfies the following conditions: (a) $\ast$ is commutative and associative, (b) $\ast$ is continuous, (c) $a \ast 1 = a$ for all $a \in [0, 1]$, (d) $a \ast b \leq c \ast d$ whenever $a \leq c$ and $b \leq d$ ($a, b, c, d \in [0, 1]$). Also, a continuous $t-$conorm is a operation $\triangledown : [0, 1] \times [0, 1] \rightarrow [0, 1]$ which satisfies the following conditions: (a) $\triangledown$ is commutative and associative, (b) $\triangledown$ is continuous, (c) $a \triangledown 0 = a$ for all $a \in [0, 1]$, (d) $a \triangledown b \geq c \triangledown d$ whenever $a \leq c$ and $b \leq d$ ($a, b, c, d \in [0, 1]$).

Definition 2.1. ([11]) The 5-tuple $(X, M, N, \ast, \triangledown)$ is said to be an intuitionistic fuzzy metric space (shortly, IFMS) if $X$ is an arbitrary set, $\ast$ is a continuous $t-$norm, $\triangledown$ is a continuous $t-$conorm and $M, N$ are fuzzy sets on $X^2 \times (0, \infty)$ satisfying the following conditions; for all $x, y \in X$ and $t > 0$, such that

(a) $M(x, y, t) > 0$,
(b) $M(x, y, t) = 1$ if and only if $x = y$,
(c) $M(x, y, t) = M(y, x, t)$,
(d) $M(x, y, t) \ast M(y, z, s) \leq M(x, z, t + s)$,
(e) $M(x, y, \cdot) : (0, \infty) \rightarrow (0, 1]$ is continuous,
(f) $N(x, y, t) > 0$,
(g) $N(x, y, t) = 0$ if and only if $x = y$,
(h) $N(x, y, t) = N(y, x, t)$,
(i) $N(x, y, t) \triangledown N(y, z, s) \geq N(x, z, t + s)$,
(j) $N(x, y, \cdot) : (0, \infty) \rightarrow (0, 1]$ is continuous.

Note that $(M, N)$ is called an intuitionistic fuzzy metric on $X$. The functions $M(x, y, t)$ and $N(x, y, t)$ denote the degree of nearness and the degree of non-nearness between $x$ and $y$ with respect to $t$, respectively.

In this paper, we use $t-$norm $\ast = \min$ and $t-$conorm $\triangledown = \max$.

Definition 2.2. ([12]) Let $X$ be an IFMS.

(a) $\{x_n\}$ is said to be convergent to a point $x \in X$ by $\lim_{n \rightarrow \infty} x_n = x$ if $\lim_{n \rightarrow \infty} M(x_n, x, t) = 1, \lim_{n \rightarrow \infty} N(x_n, x, t) = 0$ for all $t > 0$.

(b) $\{x_n\}$ is called a Cauchy sequence if for any $\epsilon > 0$, there exists $n_0 \in N$ such that for all $t > 0$ and $m, n \geq n_0$,

$M(x_n, x_m, t) > 1 - \epsilon, \quad N(x_n, x_m, t) < \epsilon$.
(c) $X$ is complete if and only if every Cauchy sequence converges in $X$.

**Definition 2.3.** Let $X$ be an IFMS. Also, let $F : X \times X \to X$, $G : X \times X \to X$, $h : X \to X$ and $g : X \to X$ be maps.

(a) An element $(x, y) \in X \times X$ is called a coupled coincidence point of a mapping $F$ and $h$ if $F(x, y) = hx$, $F(y, x) = hy$.

(b) The maps $F$ and $h$ are commuting if for all $x, y \in X$, $hF(x, y) = F(hx, hy)$.

(c) The maps $F$ and $h$ are said to be coincidentally commuting if they commute at their coupled coincidence points, that is, if $hx = F(x, y)$ and $hy = F(y, x)$, for some $(x, y) \in X \times X$, then $hF(x, y) = F(hx, hy)$ and $hF(y, x) = F(hy, hx)$.

(d) $(x, y) \in X \times X$ is called a common coupled coincidence point of the pairs of $(F, h)$ and $(G, g)$ if $F(x, y) = hx$, $F(y, x) = hy$, $(G(x, y) = gx$ and $G(y, x) = gy$.

**Example 2.4.** Let $X$ be an IFMS and let $F : X \times X \to X$ and $h : X \to X$ be defined respectively as follows;

$$F(x, y) = \begin{cases} 1 & \text{if } x > 1, \ 0 < y < 1, \\
\frac{1}{3} & \text{otherwise,}
\end{cases} \quad h(x) = \begin{cases} 0 & \text{if } x = 0, \\
100 & \text{if } 0 < x < 1, \\
1 & \text{if } x = 1, \\
20 & \text{if } x > 1.
\end{cases}$$

The maps $F, h$ commute at their only coupled coincidence point $(0, 0)$. Therefore the pair of maps $(F, h)$ is coincidentally commuting. But the pair of maps $(F, h)$ is not commuting.

**Definition 2.5.** Let $X$ be an IFMS. The maps $F : X \times X \to X$ and $h : X \to X$ are said to satisfy property (E.A.) if there exists two sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that for some $x, y \in X$,

$$F(x_n, y_n) = hx_n \to x \quad \text{as} \quad n \to \infty,$$

$$F(y_n, x_n) = hy_n \to y \quad \text{as} \quad n \to \infty.$$

In this paper, we use the following class of real maps.

Define $\Phi = \{\phi, \psi : [0, 1] \to [0, 1]\}$ satisfying the conditions:

(i) $\phi$ is continuous monotone increasing, $\psi$ is continuous monotone decreasing on $[0, 1]$,

(ii) $\phi(t) > t$ and $\psi(t) < t$ for all $0 < t < 1$ with $\phi(1) = 1$ and $\psi(0) = 0$.

3. **Main result**

**Theorem 3.1.** Let $X$ be an IFMS. Let $F : X \times X \to X$, $G : X \times X \to X$, $h : X \to X$ and $g : X \to X$ be four maps and $\phi, \psi \in \Phi$ satisfies the following conditions:'
(a) For all \( x, y, u, v \in X, \ s > 0 \) and \( \phi, \psi \in \Phi \)

\[ M(F(x, y), G(u, v), s) \]
\[ \geq \phi(\min\{M(hx, gu, s), M(hy, gv, s), M(hx, F(x, y), s), \]
\[ M(hx, G(u, v), s), M(gu, G(u, v), s), M(gu, F(x, y), s)\}) \],
\[ N(F(x, y), G(u, v), s) \]
\[ \leq \psi(\max\{N(hx, gu, s), N(hy, gv, s), N(hx, F(x, y), s), \]
\[ N(hx, G(u, v), s), N(gu, G(u, v), s), N(gu, F(x, y), s)\}) \],

(b) \( F(X \times X) \subseteq g(X), \ G(X \times X) \subseteq h(X) \) and \( h(X), g(X) \) are two closed subsets of \( X \),
(c) \( (h, F) \) and \( (g, G) \) are coincidentally commuting pairs.

If \( (h, F) \) and \( (g, G) \) satisfy the property(E.A.), then there exist \( x, y \in X \)
such that \( hx = F(x, y), hy = F(y, x), gx = G(x, y) \) and \( gy = G(y, x) \), that is, the pairs of maps \( (h, F) \) and \( (g, G) \) have common coupled coincidence point in \( X \).

Proof. Since \( (h, F) \) and \( (g, G) \) satisfy the property(E.A.), there exist two sequences \( \{x_n\} \) and \( \{y_n\} \) in \( X \), such that

\[ F(x_n, y_n) = hx_n \rightarrow x \ as \ n \rightarrow \infty, \]
\[ F(y_n, x_n) = hy_n \rightarrow y \ as \ n \rightarrow \infty. \]

and

\[ G(x_n, y_n) = gx_n \rightarrow x \ as \ n \rightarrow \infty, \]
\[ G(y_n, x_n) = gy_n \rightarrow y \ as \ n \rightarrow \infty. \]

Hence \( x, y \in h(X) \cap g(X) \) because \( h(X), g(X) \) are two closed subsets of \( X \).
Also, since \( G(X \times X) \subseteq h(X) \), there exists \( u, v \in X \) such that \( hu = x, hv = y \).

Thus for all \( s > 0 \), we have

\[ M(F(u, v), G(x_n, y_n), s) \]
\[ \geq \phi(\min\{M(hu, gx_n, s), M(hv, gy_n, s), M(hu, F(u, v), s), \]
\[ M(gx_n, G(x_n, y_n), s), M(hu, G(x_n, y_n), s), M(gx_n, F(u, v), s)\}) \],
\[ N(F(u, v), G(x_n, y_n), s) \]
\[ \leq \psi(\max\{N(hu, gx_n, s), N(hv, gy_n, s), N(hu, F(u, v), s), \]
\[ N(gx_n, G(x_n, y_n), s), N(hu, G(x_n, y_n), s), N(gx_n, F(u, v), s)\}). \]
As \( n \to \infty \), we have for all \( s > 0 \),
\[
M(F(u, v), x, s) \\
\geq \phi(\min\{M(x, x, s), M(y, y, s), M(x, F(u, v), s), M(x, x, s), M(x, F(u, v), s)\}) \\
\geq \phi(\min\{1, 1, M(x, F(u, v), s), 1, 1, M(x, F(u, v), s)\}), \\
N(F(u, v), x, s) \\
\leq \psi(\max\{N(x, x, s), N(y, y, s), N(x, F(u, v), s), N(x, x, s), N(x, F(u, v), s)\}) \\
\leq \psi(\max\{0, 0, N(x, F(u, v), s), 0, 0, N(x, F(u, v), s)\}).
\]
Now, if \( x \neq F(u, v) \), then \( M(F(u, v), x, s) \geq \phi(M(x, F(u, v), s)) > M(x, F(u, v), s) \)
and \( N(F(u, v), x, s) \leq \psi(N(x, F(u, v), s)) < N(x, F(u, v), s) \) which is a contradiction in this inequality.
Hence \( M(x, F(u, v), s) = 1 \) and \( N(x, F(u, v), s) = 0 \) which implies that \( x = F(u, v) \). Therefore \( x = hu = F(u, v) \). Similarly, we can prove that \( y = hv = F(v, u) \).
Since \( F(X \times X) \subseteq g(X) \), there exists \( w, z \in X \) such that \( gw = x, gz = y \).
Thus for all \( s > 0 \), we have
\[
M(F(x_n, y_n), G(w, z), s) \\
\geq \phi(\min\{M(hx_n, gw, s), M(hy_n, gz, s), M(hx_n, F(x_n, y_n), s), M(gw, G(w, z), s), M(hx_n, G(w, z), s), M(gw, F(x_n, y_n), s)\}), \\
N(F(x_n, y_n), G(w, z), s) \\
\leq \psi(\max\{N(hx_n, gw, s), N(hy_n, gz, s), N(hx_n, F(x_n, y_n), s), N(gw, G(w, z), s), N(hx_n, G(w, z), s), N(gw, F(x_n, y_n), s)\}).
\]
As \( n \to \infty \), we have for all \( s > 0 \),
\[
M(x, G(w, z), s) \\
\geq \phi(\min\{M(x, x, s), M(y, y, s), M(x, x, s), M(x, G(w, z), s), M(x, G(w, z), s), M(x, x, s)\}) \\
\geq \phi(\min\{1, 1, 1, M(x, G(w, z), s), M(x, G(w, z), s), 1\}), \\
N(x, G(w, z), s) \\
\leq \psi(\max\{N(x, x, s), N(y, y, s), N(x, x, s), N(x, G(w, z), s), N(x, x, s)\}) \\
\leq \psi(\max\{0, 0, 0, N(x, G(w, z), s), N(x, G(w, z), s), 0\}).
\]
Now, if \( x \neq G(w, z) \), then \( M(x, G(w, z), s) \geq \phi(M(x, G(w, z), s)) > M(x, G(w, z), s) \)
and \( N(x, G(w, z), s) \leq \psi(N(x, G(w, z), s)) < N(x, G(w, z), s) \) which is a contradiction in this inequality.
Hence \( M(x, G(w, z), s) = 1 \) and \( N(x, G(w, z), s) = 0 \) which implies that \( x = G(w, z) \). Therefore \( x = gw = G(w, z) \). Similarly, we can prove that \( y = gz = G(z, w) \). Therefore \( x = gw = hu = G(w, z) = F(u, v) \)
and \( y = gz = hv = F(v, u) = G(z, w) \). Since \((h, F)\) is coincidentally
commuting, \( hF(u,v) = F(hu, hv) \) and \( hF(v,u) = F(hv, hu) \) which implies \( hx = F(x,y) \) and \( hy = F(y,x) \). Also, since \((g,G)\) is coincidentally commuting, therefore \( gG(w,z) = G(gw,gz) \) and \( gG(z,w) = G(gz,gw) \) which implies \( gx = G(x,y) \) and \( gy = G(y,x) \), that is, \((x,y)\) is the common coupled coincidence point of the pairs of mappings \((h,F)\) and \((g,G)\). This completes the proof of the theorem.

\[\Box\]

**Corollary 3.2.** Let \( X \) be an IFMS. Let \( F : X \times X \rightarrow X \) and \( h : X \rightarrow X \) be two maps and \( \phi, \psi \in \Phi \) satisfies the following conditions;

(a)\hspace{1cm} for all \( x, y, u, v \in X \), \( s > 0 \) and \( \phi, \psi \in \Phi \)

\[
M(F(x, y), F(u, v), s) \\
\geq \phi(\min \{M(hx, hu, s), M(hy, hv, s), M(hx, F(x, y), s), \\
M(hu, F(u, v), s), M(hx, F(u, v), s), M(hu, F(x, y), s)\}) \\
N(F(x, y), F(u, v), s) \\
\leq \psi(\max \{N(hx, hu, s), N(hy, hv, s), N(hx, F(x, y), s), \\
N(hu, F(u, v), s), N(hx, F(u, v), s), N(hu, F(x, y), s)\})
\]

(b)\hspace{1cm} \( F(X \times X) \subseteq h(X) \) and \( h(X) \) is a closed subsets of \( X \),

(c)\hspace{1cm} \( (h,F) \) is coincidentally commuting pair.

If \((h,F)\) satisfy the property\((E.A.)\), then there exist \( x, y \in X \) such that \( hx = F(x,y) \) and \( hy = F(y,x) \), that is, \((h,F)\) has coupled coincidence point in \( X \).

**Proof.** The proof follows by putting \( F = G, h = g \) in Theorem 3.1.

\[\Box\]

**Corollary 3.3.** Let \( X \) be an IFMS. Let \( F : X \times X \rightarrow X \) be a map and \( \phi, \psi \in \Phi \) satisfies the following conditions;

For all \( x, y, u, v \in X, s > 0 \) and \( \phi, \psi \in \Phi \)

\[
M(F(x, y), F(u, v), s) \\
\geq \phi(\min \{M(x, u, s), M(y, v, s), M(x, F(x, y), s), \\
M(u, F(u, v), s), M(x, F(u, v), s), M(u, F(x, y), s)\}) \\
N(F(x, y), F(u, v), s) \\
\leq \psi(\max \{N(x, u, s), N(y, v, s), N(x, F(x, y), s), \\
N(u, F(u, v), s), N(x, F(u, v), s), N(u, F(x, y), s)\})
\]

If \( F \) satisfy the property\((E.A.)\), then there exist \( x, y \in X \) such that \( x = F(x,y) \) and \( y = F(y,x) \), that is, \( F \) has fixed point in \( X \).

**Proof.** The proof follows by putting \( F = G, h = g = I\)(Identity function) in Theorem 3.1.

\[\Box\]

**References**