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Reducibility of the Kampé de Fériet Function

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Abstract

It has been an interesting and natural research subject to consider the reducibility of some extensively generalized special functions. In this regard, the Kampé de Fériet function has attracted many mathematicians to study its properties. The authors [9] also established many interesting cases of the reducibility of the Kampé de Fériet function by employing generalizations of the two results for the terminating ${}_2F_1(2)$ hypergeometric identities due to Kim *et al.* [18]. In this sequel, we aim at presenting several interesting cases of the reducibility of Kampé de Fériet function by using generalizations of classical Kummer's summation theorem due to Choi [8]. Well-known results due to Bailey, Saran, Kim *et al.* and other contiguous results are pointed to be obtained as some special cases of our main findings.

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1 Introduction and Preliminaries

Throughout this paper, $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{Z}_0^-$ denote the sets of positive integers, real numbers, complex numbers, and nonpositive integers, respectively, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The generalized hypergeometric series ${}_pF_q$ is defined by (see [26, p. 73]; see also [3, 31]):

$$\begin{aligned} {}_pF_q\left[\begin{matrix} \alpha_1, \ldots, \alpha_p; \\ \beta_1, \ldots, \beta_q; \end{matrix} z\right] &= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!} \\ &= {}_pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z), \end{aligned} \tag{1}$$

where $(\lambda)_n$ is the Pochhammer symbol defined (for $\lambda \in \mathbb{C}$) by (see [32, p. 2 and p. 5]):

$$\begin{aligned} (\lambda)_n := &\begin{cases} 1 & (n = 0) \\ \lambda(\lambda+1)\ldots(\lambda+n-1) & (n \in \mathbb{N}) \end{cases} \\ = &\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \quad (\lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-) \end{aligned} \tag{2}$$

and $\Gamma(\lambda)$ is the familiar Gamma function. Here p and q are positive integers or zero (interpreting an empty product as 1), and we assume (for simplicity) that the variable z, the numerator parameters $\alpha_1, \ldots, \alpha_p$, and the denominator parameters $\beta_1, \ldots, \beta_q$ take on complex values, provided that no zeros appear in the denominator of (1), that is, that

$$(\beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ j = 1, \ldots, q). \tag{3}$$

Thus, if a numerator parameter is a negative integer or zero, the ${}_pF_q$ series terminates in view of the identity (see [32, p. 5]):

$$(-n)_k = \begin{cases} \dfrac{(-1)^k\, n!}{(n-k)!} & (0 \le k \le n;\ n \in \mathbb{N}), \\ 0 & (k > n). \end{cases} \tag{4}$$

It should be remarked here that whenever hypergeometric and generalized hypergeometric functions are expressed in terms of the Gamma function, the results are usually important from the applications point of view. Therefore, the well known summation theorems such as those of Gauss, Gauss's second, Bailey and Kummer for the series ${}_2F_1$ and Watson, Dixon and Whipple for the series ${}_3F_2$ and their extensions and generalizations (see [22], [23], [24], [25]) play an important role in the theory of generalized hypergeometric series. For applications of the above-mentioned classical summation theorems, we refer to [2], [17], [18], [25], [26], [27].

Recently a good deal of research has been done in the direction of generalizing and extending the above mentioned classical summation theorems (see, *e.g.*, [17, 19, 22, 23, 24, 25, 36]; for applications, [2, 15, 16, 18, 20]). It is also well known that, if the product of two (generalized) hypergeometric series can be expressed as a (generalized) hypergeometric series with argument x, the coefficient of x^n in the product must be expressible in terms of Gamma functions. In this regard, we recall the following very interesting result due to Bailey [2]:

$$ {}_0F_1\left[\begin{matrix} -\,; \\ e\,; \end{matrix}\; x\right] \times {}_0F_1\left[\begin{matrix} -\,; \\ e\,; \end{matrix}\; -x\right] = {}_0F_3\left[\begin{matrix} \overline{\qquad\quad}\,; \\ e,\ \frac{1}{2}e,\ \frac{1}{2}e+\frac{1}{2}\,; \end{matrix}\; -\frac{x^2}{4}\right]. \tag{5} $$

Bailey [2] obtained the result (5) by making use of the following classical Kummer's summation theorem:

$$ {}_2F_1\left[\begin{matrix} a,\ b\,; \\ 1+a-b\,; \end{matrix}\; -1\right] = \frac{\Gamma\left(1+\frac{1}{2}a\right)\,\Gamma\left(1+a-b\right)}{\Gamma\left(1+a\right)\,\Gamma\left(1+\frac{1}{2}a-b\right)}. \tag{6} $$

Lavoie *et al.* [24] generalized the classical Kummer's summation theorem (6) in the form

$$ {}_2F_1\left[\begin{matrix} a,\ b\,; \\ 1+a-b+i\,; \end{matrix}\; -1\right] \quad (i = 0,\ \pm1,\ \ldots,\ \pm5), $$

which was further extended for $i = 0,\ \pm1,\ \ldots,\ \pm9$ by Choi [8] as follows:

$$ \begin{aligned} {}_2F_1\left[\begin{matrix} a,\ b\,; \\ 1+a-b+i\,; \end{matrix}\; -1\right] &= \frac{2^{-a}\,\Gamma\left(\frac{1}{2}\right)\,\Gamma\left(1-b\right)\,\Gamma\left(1+a-b+i\right)}{\Gamma\left(1-b+\frac{1}{2}(i+|i|)\right)} \\ &\times\left\{\frac{\mathcal{A}_i(a,b)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}i+\frac{1}{2}-\left[\frac{i+1}{2}\right]\right)\,\Gamma\left(1+\frac{1}{2}a-b+\frac{1}{2}i\right)}\right. \\ &\quad\left. + \frac{\mathcal{B}_i(a,b)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}i-\left[\frac{i}{2}\right]\right)\,\Gamma\left(\frac{1}{2}+\frac{1}{2}a-b+\frac{1}{2}i\right)}\right\}, \end{aligned} \tag{7} $$

where $i = 0,\ \pm1,\ \ldots,\ \pm9$. Here, as usual, $[x]$ denotes the greatest integer less than or equal to $x \in \mathbb{R}$ and its absolute value is denoted by $|x|$. The coefficients $\mathcal{A}_i(a,b) := \mathcal{A}_i$ and $\mathcal{B}_i(a,b) := \mathcal{B}_i$ are given in the following tables.

It is interesting to mention here that the vast popularity and immense usefulness of the hypergeometric function ${}_2F_1$ and the generalized hypergeometric functions ${}_pF_q$ ($p,\ q \in \mathbb{N}_0$) in one variable have inspired and stimulated a large number of research workers to investigate hypergeometric functions of two or more variables. Serious and significant study of the functions of two variables was initiated by Appell [1] who presented the so-called Appell functions F_1, F_2, F_3 and F_4 which are natural generalizations of the Gaussian hypergeometric function and whose confluent forms were studied by Humbert [33, 34]. A complete list of these functions can be seen in the standard text of Erdélyi *et al.* [10]. Also, later on, the four Appell functions F_1, F_2, F_3 and F_4 and their confluent forms were further generalized by Kampé de Fériet [1], who introduced a more general hypergeometric function of two variables. The notation defined and introduced by the Kampé de Fériet for his double hypergeometric function of superior order was subsequently abbreviated by Burchnall and Chaundy [4, 5]. We, however, recall here the definition of a more general double hypergeometric function (than the one defined by the Kampé de Fériet) in a slightly modified notation given by Srivastava and Panda [35, p. 423, Eq.(26)]. Therefore, the most convenient generalization of the Kampé de Fériet is defined as follows:

$$\begin{aligned} & F^{H:\,A;\,B}_{G:\,C;\,D}\left[\begin{matrix}(h_H)\ :\ (a_A)\ ;\ (b_B);\\ (g_G)\ :\ (c_C)\ ;\ (d_D);\end{matrix}\ x,\, y\right] \\ & \quad = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{((h_H))_{m+n}\ ((a_A))_m\ ((b_B))_n}{((g_G))_{m+n}\ ((c_C))_m\ ((d_D))_n}\,\frac{x^m}{m!}\,\frac{y^n}{n!}, \end{aligned} \tag{8}$$

where (h_H) denotes the sequence of parameters $(h_1,\, h_2,\, \ldots,\, h_H)$ and, for $n \in \mathbb{N}_0$, define the Pochhammer symbol

$$((h_H))_n := (h_1)_n \cdots (h_H)_n,$$

where, when $n = 0$, the product is understood to reduce to unity. The symbol (h) is a convenient contraction for the sequence of the parameters $h_1,\, h_2,\, \ldots,$ h_H and the Pochhammer symbol $(h)_n$ is the same as defined in (2). For details about the convergence for this function, we refer to [33].

It has been an interesting and natural research subject to consider the reducibility of some extensively generalized special functions. In this regard, the Kampé de Fériet function has attracted many mathematicians to study its properties (see, *e.g.*, [6, 9, 12, 13, 14, 21, 28, 29]). The authors [9] also established many interesting cases of the reducibility of the Kampé de Fériet function by employing generalizations of the two results for the terminating ${}_2F_1(2)$ hypergeometric identities due to Kim *et al.* [19]. In this sequel, we aim at presenting several interesting cases of the reducibility of Kampé de Fériet function by using generalizations of classical Kummer's summation theorem due to Choi [8]. Well-known results due to Bailey [2], Saran [28], Kim *et al.* [19] and other contiguous results are obtained as some special cases of our main findings. The results presented here are simple, interesting, easily derivable, and (potentially) useful.

2 Main Results

We establish a general formula for the reducibility of the Kampé de Fériet function which is expressed in a single form containing eleven results asserted by the following theorem.

Theorem. *The following reducibility of the Kampé de Fériet function holds true:*

$$F^{D:\,0;\,0}_{G:\,1;\,1}\left[\begin{matrix}(d) & : & \text{——}; & \text{——}; & \\ (g) & : & p+i\ ; & p\ \ ; & -x,\ x\end{matrix}\right] = \frac{\Gamma\left(\frac{1}{2}\right)\,\Gamma\left(p\right)\,\Gamma\left(p+i\right)}{\Gamma\left(p+\frac{1}{2}(i+|i|)\right)} \tag{9}$$

$$\times \sum_{n=0}^{\infty} \frac{4^{n(D-G-1)}\,(-x^2)^n\,((\frac{1}{2}d))_n\,((\frac{1}{2}d)+\frac{1}{2})_n}{n!\,((\frac{1}{2}g))_n\,((\frac{1}{2}g)+\frac{1}{2})_n\,(\frac{1}{2})_n\,(\frac{1}{2}p+\frac{1}{4}(i+|i|))_n\,(\frac{1}{2}p+\frac{1}{4}(i+|i|)+\frac{1}{2})_n}$$

$$\times \left\{ \frac{\mathcal{A}_i'\,(\frac{1}{2}-\frac{1}{2}i+[\frac{1+i}{2}])_n}{\Gamma(p+\frac{1}{2}i)\,\Gamma(\frac{1}{2}i+\frac{1}{2}-[\frac{1+i}{2}])\,(p+\frac{1}{2}i)_n} + \frac{\mathcal{B}_i'\,(1-\frac{1}{2}i+[\frac{i}{2}])_n}{\Gamma(p+\frac{1}{2}i-\frac{1}{2})\,\Gamma(\frac{1}{2}i-[\frac{1}{2}i])\,(p+\frac{1}{2}i-\frac{1}{2})_n} \right\}$$

$$+\frac{(d)}{(g)}\,2x\,\frac{\Gamma\left(\frac{1}{2}\right)\,\Gamma\left(p\right)\,\Gamma\left(p+i\right)}{\Gamma\left(1+p+\frac{1}{2}(i+|i|)\right)}$$

$$\times \sum_{n=0}^{\infty} \frac{4^{n(D-G-1)}\,(-x^2)^n\,((\frac{1}{2}d)+\frac{1}{2})_n\,((\frac{1}{2}d+\frac{1}{2})+\frac{1}{2})_n}{n!\,((\frac{1}{2}g+\frac{1}{2}))_n\,((\frac{1}{2}g+\frac{1}{2})+\frac{1}{2})_n\,(\frac{3}{2})_n\,(\frac{1}{2}+\frac{1}{2}p+\frac{1}{4}(i+|i|))_n\,(1+\frac{1}{2}p+\frac{1}{4}(i+|i|))_n}$$

$$\times \left\{ \frac{\mathcal{A}_i''\,(1-\frac{1}{2}i+[\frac{1+i}{2}])_n}{\Gamma(\frac{1}{2}+\frac{1}{2}i+p)\,\Gamma(\frac{1}{2}i-[\frac{1+i}{2}])\,(\frac{1}{2}+\frac{1}{2}i+p)_n} + \frac{\mathcal{B}_i''\,(\frac{3}{2}-\frac{1}{2}i+[\frac{i}{2}])_n}{\Gamma(p+\frac{1}{2}i)\,\Gamma(\frac{1}{2}i-[\frac{1}{2}i]-\frac{1}{2})\,(p+\frac{1}{2}i)_n} \right\},$$

where $i = 0, \pm 1, \ldots, \pm 9$. *Here, as usual,* $[x]$ *denotes the greatest integer less than or equal to* $x \in \mathbb{R}$ *and its absolute value is denoted by* $|x|$. *The coefficients* $\mathcal{A}_i'$ *and* $\mathcal{B}_i'$ *can be obtained from the* Table *of* $\mathcal{A}_i$ and $\mathcal{B}_i$ *by simply substituting* a *and* b *with* $-2n$ and $1-p-2n$, *respectively, while the coefficients* $\mathcal{A}_i''$ *and* $\mathcal{B}_i''$ *can be obtained from the* Table *of* $\mathcal{A}_i$ *and* $\mathcal{B}_i$ *by substituting* a *and* b *with* $-2n-1$ *and* $-p-2n$, *respectively.*

Proof. For simplicity and convenience, we start with denoting the left-hand side of (9) by $\mathcal{S}$. Expressing the Kampé de Fériet function in double series, we have

$$\mathcal{S} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{((d))_{m+n}}{((g))_{m+n}} \frac{(-1)^m\,x^{m+n}}{(p+i)_m\,(p)_n\,m!\,n!}.$$

Replacing n by $n-m$ and using a well-known and easily-verified double series manipulation (see, *e.g.*, [7]):

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A(m,n) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} A(m,n-m),$$

we obtain

$$\mathcal{S}=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{((d))_n}{((g))_n}\frac{(-1)^m\,x^n}{(p+i)_m\,(p)_{n-m}\,m!\,(n-m)!}.$$

Using the elementary identities (see, e.g., [32, p. 5]:

$$(\lambda)_{n-m}=\frac{(-1)^m\,(\lambda)_n}{(1-\lambda-n)_m}\quad (0\leqq m\leqq n)$$

and

$$(n-m)!=\frac{(-1)^m\,n!}{(-n)_m}\quad (0\leqq m\leqq n),$$

after a little simplification, we get

$$\mathcal{S}=\sum_{n=0}^{\infty}\frac{((d))_n}{((g))_n}\frac{x^n}{(p)_n\,n!}\sum_{m=0}^{n}\frac{(-n)_m\,(1-p-n)_m\,(-1)^m}{(p+i)_m\,m!}.$$

Expressing the inner sum as a ${}_2F_1$, we have

$$\mathcal{S}=\sum_{n=0}^{\infty}\frac{((d))_n}{((g))_n}\frac{x^n}{(p)_n\,n!}\,{}_2F_1\left[\begin{matrix}-n,\ 1-p-n\,; \\ p+i\,;\end{matrix}\ -1\right].$$

Separating the final summation into even and odd powers of x, we obtain

$$\begin{aligned}\mathcal{S}=&\sum_{n=0}^{\infty}\frac{((d))_{2n}}{((g))_{2n}}\frac{x^{2n}}{(p)_{2n}\,(2n)!}\,{}_2F_1\left[\begin{matrix}-2n,\ 1-p-2n\,; \\ p+i\,;\end{matrix}\ -1\right]\\ &+\sum_{n=0}^{\infty}\frac{((d))_{2n+1}}{((g))_{2n+1}}\frac{x^{2n+1}}{(p)_{2n+1}\,(2n+1)!}\,{}_2F_1\left[\begin{matrix}-2n-1,\ -p-2n\,; \\ p+i\,;\end{matrix}\ -1\right].\end{aligned}$$

Finally, evaluating both ${}_2F_1$ with the help of (7) and making use of the following identity (see, e.g., [32, p. 6]:

$$(\lambda)_{2n}=2^{2n}\left(\frac{1}{2}\lambda\right)_n\left(\frac{1}{2}\lambda+\frac{1}{2}\right)_n\quad (n\in\mathbb{N}_0),$$

after some algebra, we arrive at the right-hand side of our general formula (9). This completes the proof of (9). □

3 Some results derivable from (9)

Here we shall mention some interesting known as well as new results in compact forms. By setting $i=0,\ \pm1,\ \pm2$ in our main formula (9) and summing up the resulting series on the right-hand side, we get we get some interesting and (potentially) useful results in compact forms given by the following corollary.

Corollary. *Each of the following formulas holds true.*

$$F_{G:\,1;\,1}^{D:\,0;\,0}\left[\begin{matrix}(d) & : & \text{---}; & \text{---}; \\ (g) & : & p\;; & p\;;\end{matrix}\; -x,\, x\right]$$
$$= {}_{2D}F_{2G+3}\left[\begin{matrix} \left(\frac{1}{2}d\right),\ \left(\frac{1}{2}d\right)+\frac{1}{2}\,; \\ \left(\frac{1}{2}g\right),\ \left(\frac{1}{2}g\right)+\frac{1}{2},\ p,\ \frac{1}{2}p,\ \frac{1}{2}p+\frac{1}{2}\,; \end{matrix}\; -4^{D-G-1}\,x^2\right]. \tag{10}$$

$$F_{G:\,1;\,1}^{D:\,0;\,0}\left[\begin{matrix}(d) & : & \text{---}; & \text{---}; \\ (g) & : & p+1\;; & p\;;\end{matrix}\; -x,\, x\right]$$
$$= {}_{2D}F_{2G+3}\left[\begin{matrix} \left(\frac{1}{2}d\right),\ \left(\frac{1}{2}d\right)+\frac{1}{2}\,; \\ \left(\frac{1}{2}g\right),\ \left(\frac{1}{2}g\right)+\frac{1}{2},\ p,\ \frac{1}{2}p+\frac{1}{2},\ \frac{1}{2}p+1\,; \end{matrix}\; -4^{D-G-1}\,x^2\right]$$
$$+ \frac{(d)}{(g)}\frac{x}{p(p+1)}$$
$$\times {}_{2D}F_{2G+3}\left[\begin{matrix} \left(\frac{1}{2}d\right)+\frac{1}{2},\ \left(\frac{1}{2}d++\frac{1}{2}\right)+\frac{1}{2}\,; \\ \left(\frac{1}{2}g\right)+\frac{1}{2},\ \left(\frac{1}{2}g+\frac{1}{2}\right)+\frac{1}{2},\ p+1,\ \frac{1}{2}p+1,\ \frac{1}{2}p+\frac{3}{2}\,; \end{matrix}\; -4^{D-G-1}\,x^2\right]. \tag{11}$$

$$F_{G:\,1;\,1}^{D:\,0;\,0}\left[\begin{matrix}(d) & : & \text{---}; & \text{---}; \\ (g) & : & p-1\;; & p\;;\end{matrix}\; -x,\, x\right]$$
$$= {}_{2D}F_{2G+3}\left[\begin{matrix} \left(\frac{1}{2}d\right),\ \left(\frac{1}{2}d\right)+\frac{1}{2}\,; \\ \left(\frac{1}{2}g\right),\ \left(\frac{1}{2}g\right)+\frac{1}{2},\ p-1,\ \frac{1}{2}p,\ \frac{1}{2}p+\frac{1}{2}\,; \end{matrix}\; -4^{D-G-1}\,x^2\right]$$
$$- \frac{(d)}{(g)}\frac{x}{p(p-1)}$$
$$\times {}_{2D}F_{2G+3}\left[\begin{matrix} \left(\frac{1}{2}d+\frac{1}{2}\right),\ \left(\frac{1}{2}d+\frac{1}{2}\right)+\frac{1}{2}\,; \\ \left(\frac{1}{2}g+\frac{1}{2}\right),\ \left(\frac{1}{2}g+\frac{1}{2}\right)+\frac{1}{2},\ p,\ \frac{1}{2}p+\frac{1}{2},\ \frac{1}{2}p+1\,; \end{matrix}\; -4^{D-G-1}\,x^2\right]. \tag{12}$$

$$F^{D:\,0;\,0}_{G:\,1;\,1}\left[\begin{matrix}(d) & : & \text{——}\,; & \text{——}\,; \\ (g) & : & p+2\,; & p\,;\end{matrix}\; -x,\, x\right]$$
$$= {}_{2D}F_{2G+3}\left[\begin{matrix}\left(\frac{1}{2}d\right),\, \left(\frac{1}{2}d\right)+\frac{1}{2}\,; \\ \left(\frac{1}{2}g\right),\, \left(\frac{1}{2}g\right)+\frac{1}{2},\, p+1,\, \frac{1}{2}p+1,\, \frac{1}{2}p+\frac{3}{2}\,;\end{matrix}\; -4^{D-G-1}\,x^2\right]$$
$$+\frac{(d)}{(g)}\frac{2x}{p(p+2)}$$
$$\times {}_{2D}F_{2G+3}\left[\begin{matrix}\left(\frac{1}{2}d+\frac{1}{2}\right),\, \left(\frac{1}{2}d+\frac{1}{2}\right)+\frac{1}{2}\,; \\ \left(\frac{1}{2}g+\frac{1}{2}\right),\, \left(\frac{1}{2}g+\frac{1}{2}\right)+\frac{1}{2},\, p+1,\, \frac{1}{2}p+\frac{3}{2},\, \frac{1}{2}p+2\,;\end{matrix}\; -4^{D-G-1}\,x^2\right]. \tag{13}$$

$$F^{D:\,0;\,0}_{G:\,1;\,1}\left[\begin{matrix}(d) & : & \text{——}\,; & \text{——}\,; \\ (g) & : & p-2\,; & p\,;\end{matrix}\; -x,\, x\right]$$
$$= {}_{2D}F_{2G+3}\left[\begin{matrix}\left(\frac{1}{2}d\right),\, \left(\frac{1}{2}d\right)+\frac{1}{2}\,; \\ \left(\frac{1}{2}g\right),\, \left(\frac{1}{2}g\right)+\frac{1}{2},\, p-1,\, \frac{1}{2}p,\, \frac{1}{2}p+\frac{1}{2}\,;\end{matrix}\; -4^{D-G-1}\,x^2\right]$$
$$-\frac{(d)}{(g)}\frac{2x}{p(p-2)}$$
$$\times {}_{2D}F_{2G+3}\left[\begin{matrix}\left(\frac{1}{2}d\right)+\frac{1}{2},\, \left(\frac{1}{2}d+\frac{1}{2}\right)+\frac{1}{2}\,; \\ \left(\frac{1}{2}g\right)+\frac{1}{2},\, \left(\frac{1}{2}g+\frac{1}{2}\right)+\frac{1}{2},\, p-1,\, \frac{1}{2}p+\frac{1}{2},\, \frac{1}{2}p+1\,;\end{matrix}\; -4^{D-G-1}\,x^2\right]. \tag{14}$$

It is remarked in passing that the result (10) is a known result due to Saran [28] while the results (11) to (14) which are closely related to (10) are (presumably) new.

4 Special Cases

Here we consider some special cases of our main result (9). Setting $D=G=0$ and $i=0$ in (9) and using the definition of the Kampé de Fériet function (8), we get

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{(p)_m\,(p)_n}\frac{x^{m+n}}{m!\,n!}=\sum_{n=0}^{\infty}\frac{(-1)^n}{(p)_n\,\left(\frac{1}{2}p\right)_n\,\left(\frac{1}{2}p+\frac{1}{2}\right)_n}\frac{x^{2n}}{2^{2n}}. \tag{15}$$

Since the double series in the left-hand side of (15) can be separated in two independent series, it is easy to see that (15) is equal to the following formula:

$$ {}_0F_1\left[\begin{matrix} -\,; \\ p\,; \end{matrix}\; x\right] \times {}_0F_1\left[\begin{matrix} -\,; \\ p\,; \end{matrix}\; -x\right] = {}_0F_3\left[\begin{matrix} \overline{\qquad\qquad} \,; \\ p,\ \frac{1}{2}p,\ \frac{1}{2}p+\frac{1}{2}\,; \end{matrix}\; -\frac{x^2}{4}\right]. \tag{16} $$

Similarly, the special cases of (9) when $D = G = 0$ and $i = \pm 1, \pm 2$ are easily seen to yield the following respective product formulas:

$$ \begin{aligned} {}_0F_1\left[\begin{matrix} -\,; \\ p\,; \end{matrix}\; x\right] \times {}_0F_1\left[\begin{matrix} \overline{\quad}\,; \\ p+1\,; \end{matrix}\; -x\right] &= {}_0F_3\left[\begin{matrix} \overline{\qquad\qquad} \,; \\ p,\ \frac{1}{2}p+\frac{1}{2},\ \frac{1}{2}p+1\,; \end{matrix}\; -\frac{x^2}{4}\right] \\ &+ \frac{x}{p(p+1)}\,{}_0F_3\left[\begin{matrix} \overline{\qquad\qquad} \,; \\ p+1,\ \frac{1}{2}p+1,\ \frac{1}{2}p+\frac{3}{2}\,; \end{matrix}\; -\frac{x^2}{4}\right]. \end{aligned} \tag{17} $$

$$ \begin{aligned} {}_0F_1\left[\begin{matrix} -\,; \\ p\,; \end{matrix}\; x\right] \times {}_0F_1\left[\begin{matrix} \overline{\quad}\,; \\ p-1\,; \end{matrix}\; -x\right] &= {}_0F_3\left[\begin{matrix} \overline{\qquad\qquad} \,; \\ p-1,\ \frac{1}{2}p,\ \frac{1}{2}p+\frac{1}{2}\,; \end{matrix}\; -\frac{x^2}{4}\right] \\ &- \frac{x}{p(p-1)}\,{}_0F_3\left[\begin{matrix} \overline{\qquad\qquad} \,; \\ p,\ \frac{1}{2}p+\frac{1}{2},\ \frac{1}{2}p+1\,; \end{matrix}\; -\frac{x^2}{4}\right]. \end{aligned} \tag{18} $$

$$ \begin{aligned} {}_0F_1\left[\begin{matrix} -\,; \\ p\,; \end{matrix}\; x\right] \times {}_0F_1\left[\begin{matrix} \overline{\quad}\,; \\ p+2\,; \end{matrix}\; -x\right] &= {}_0F_3\left[\begin{matrix} \overline{\qquad\qquad} \,; \\ p,\ \frac{1}{2}p+1,\ \frac{1}{2}p+\frac{3}{2}\,; \end{matrix}\; -\frac{x^2}{4}\right] \\ &+ \frac{2x}{p(p+2)}\,{}_0F_3\left[\begin{matrix} \overline{\qquad\qquad} \,; \\ p+1,\ \frac{1}{2}p+\frac{3}{2},\ \frac{1}{2}p+2\,; \end{matrix}\; -\frac{x^2}{4}\right]. \end{aligned} \tag{19} $$

$$ \begin{aligned} {}_0F_1\left[\begin{matrix} -\,; \\ p\,; \end{matrix}\; x\right] \times {}_0F_1\left[\begin{matrix} \overline{\quad}\,; \\ p-2\,; \end{matrix}\; -x\right] &= {}_0F_3\left[\begin{matrix} \overline{\qquad\qquad} \,; \\ p-1,\ \frac{1}{2}p,\ \frac{1}{2}p+\frac{1}{2}\,; \end{matrix}\; -\frac{x^2}{4}\right] \\ &- \frac{2x}{p(p-2)}\,{}_0F_3\left[\begin{matrix} \overline{\qquad\qquad} \,; \\ p-1,\ \frac{1}{2}p+\frac{1}{2},\ \frac{1}{2}p+1\,; \end{matrix}\; -\frac{x^2}{4}\right]. \end{aligned} \tag{20} $$

It is noted that (16) is a known result due to Bailey [2] (see also (5)) and the results (17) to (20) which are closely related to the Bailey's one (16) are obtained earlier by Kim and Rathie [19] who used a different method from the one here.

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Table 1: Table for $\mathcal{A}_i$ and $\mathcal{B}_i$

i	$\mathcal{A}_i$	$\mathcal{B}_i$
9	$-16a^4+72a^3b-108a^2b^2+60ab^3$ $-9b^4-328a^3+972a^2b-792ab^2$ $+150b^3-2240a^2+3612ab-999b^2$ $-5696a+3162b-3984$	$16a^4-56a^3b+60a^2b^2-20ab^3+b^4$ $+248a^3-516a^2b+240ab^2-10b^3$ $+1160a^2-1028ab+35b^2$ $+1576a-50b-24$
8	$8a^4-32a^3b+40a^2b^2-16ab^3+b^4$ $+128a^3-312a^2b+176ab^2-10b^3+624a^2$ $+624a^2-672ab+35b^2$ $+896a-50b+24$	$8b^3-40ab^2+48a^2b-16a^3-192a^2$ $+312ab-88b^2-640a+352b-512$
7	$7b^3-28ab^2+28a^2b-8a^3-100a^2$ $+196ab-70b^2-352a+245b-302$	$8a^3-20a^2b+12ab^2-b^3+68a^2$ $-76ab+6b^2+128a-11b+6$
6	$4a^3-12a^2b+9ab^2-b^3+36a^2-51ab$ $+6b^2+74a-11b+6$	$16ab-8a^2-6b^2-48a+34b-52$
5	$10ab-4a^2-5b^2-26a+25b-32$	$4a^2-6ab+b^2+14a-3b+2$
4	$2a^2-4ab+b^2+8a-3b+2$	$4(b-a-2)$
3	$3b-2a-5$	$2a-b+1$
2	$1+a-b$	-2
1	-1	1
0	1	0

Table 2: Table for $\mathcal{A}_i$ and $\mathcal{B}_i$

i	$\mathcal{A}_i$	$\mathcal{B}_i$
-9	$16a^4-72a^3b+108a^2b^2-60ab^3$ $+9b^4-320a^3+972a^2b-828ab^2$ $+174b^3+2240a^2-3936ab+1323b^2$ $-6400a+4614b+6144$	$16a^4-56a^3b+60a^2b^2-20ab^3+b^4$ $-256a^3+564a^2b-300ab^2+26b^3$ $+1376a^2-1568ab+251b^2$ $-2816a+1066b+1680$
-8	$8a^4-32a^3b+40a^2b^2-16ab^3+b^4$ $-128a^3+328a^2b-208ab^2+22b^3+688a^2$ $-928ab+179b^2-1408a+638b+840$	$16a^3-48a^2b+40ab^2-8b^3-192a^2$ $+328ab-104b^2+704a-480b-768$
-7	$8a^3-28a^2b+28ab^2-7b^3-96a^2$ $+196ab-77b^2+352a-294b-384$	$8a^3-20a^2b+12ab^2-b^3-72a^2$ $+92ab-15b^2+184a-74b-120$
-6	$4a^3-12a^2b+9ab^2-b^3-36a^2+57ab$ $-12b^2+92a-47b-60$	$8a^2-16ab+6b^2-48a+38b+64$
-5	$4a^2-10ab+5b^2-24a+25b+32$	$4a^2-6ab+b^2-16a+7b+12$
-4	$2a^2-4ab+b^2-8a+5b+6$	$4(a-b-2)$
-3	$2a-3b-4$	$2a-b-2$
-2	$a-b-1$	2
-1	1	1