Symmetric Properties for Carlitz’s Twisted $q$-Euler Numbers and Polynomials Associated with $p$-Adic Integral on $\mathbb{Z}_p$

C. S. Ryoo

Department of Mathematics
Hannam University, Daejeon 306-791, Korea

Abstract

In this paper, we obtain some interesting symmetric identities for Carlitz’s twisted $q$-Euler polynomials in $p$-adic field. Some interesting results and relationships are obtained.

Mathematics Subject Classification: 11B68, 11S40, 11S80

Keywords: Euler numbers and polynomials, $q$-Euler numbers and polynomials, twisted $q$-Euler numbers and polynomials, symmetric properties, $p$-adic invariant integral on $\mathbb{Z}_p$

1 Introduction

The Euler numbers and polynomials possess many interesting properties and arising in many areas of mathematics and physics. Many mathematicians have studied in the area of the extension of Bernoulli numbers and polynomials, Euler numbers and polynomials, and tangent numbers and polynomials([1-11]). Recently, Y. Hu studied several identities of symmetry for Carlitz’s $q$-Bernoulli numbers and polynomials in complex field(see [1]). D. Kim et al.[4] derived some identities of symmetry for Carlitz’s $q$-Euler numbers and polynomials in complex field. In this paper, we establish some interesting symmetric identities for Carlitz’s twisted $q$-Euler polynomials in $p$-adic field. Throughout this paper
we use the following notations. By \( \mathbb{Z}_p \) we denote the ring of \( p \)-adic rational integers, \( \mathbb{Q} \) denotes the field of rational numbers, \( \mathbb{Q}_p \) denotes the field of \( p \)-adic rational numbers, \( \mathbb{C} \) denotes the complex number field, and \( \mathbb{C}_p \) denotes the completion of algebraic closure of \( \mathbb{Q}_p \). Let \( \nu_p \) be the normalized exponential valuation of \( \mathbb{C}_p \) with \( |p|_p = p^{-\nu_p(p)} = p^{-1} \). When one talks of \( q \)-extension, \( q \) is considered in many ways such as an indeterminate, a complex number \( q \in \mathbb{C} \), or \( p \)-adic number \( q \in \mathbb{C}_p \). If \( q \in \mathbb{C} \) one normally assume that \( |q| < 1 \). If \( q \in \mathbb{C}_p \), we normally assume that \( |q - 1|_p < p^{1 - \nu_p(q)} \) so that \( q^x = \exp(x \log q) \) for \( |x|_p \leq 1 \). Let \( UD(\mathbb{Z}_p) \) be the space of uniformly differentiable function on \( \mathbb{Z}_p \). Let \( g \in UD(\mathbb{Z}_p) \) be the set of uniformly differentiable function on \( \mathbb{Z}_p \). For \( g \in UD(\mathbb{Z}_p) \), Kim defined the the \( p \)-adic invariant integral on \( \mathbb{Z}_p \) as follows:

\[
I_{-1}(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x) = \lim_{N \to \infty} \sum_{x=0}^{pN-1} g(x)(-1)^x, \quad \text{see } [2, 3, 4, 5]. \tag{1.1}
\]

Let

\[
T_p = \bigcup_{m \geq 1} C_{p^m} = \lim_{m \to \infty} C_{p^m},
\]

where \( C_{p^m} = \{ \zeta | \zeta^{p^m} = 1 \} \) is the cyclic group of order \( p^m \). For \( \zeta \in T_p \), we denote by \( \phi_\zeta : \mathbb{Z}_p \to \mathbb{C}_p \) the locally constant function \( x \mapsto \zeta^x \) (see [8, 9, 10, 11]).

## 2 Symmetric identities for Carlitz’s twisted \( q \)-Euler numbers and polynomials

Our primary goal of this section is to obtain symmetric identities for Carlitz’s twisted \( q \)-Euler numbers \( \mathcal{E}_{n,q,\zeta} \) and polynomials \( \mathcal{E}_{n,q,\zeta}(x) \).

For \( q \in \mathbb{C}_p \) with \( |q - 1|_p < 1 \), twisted \( q \)-Euler polynomials \( \mathcal{E}_{n,q,\zeta}(x) \) are defined by

\[
\mathcal{E}_{n,q,\zeta}(x) = \int_{\mathbb{Z}_p} \phi_\zeta(y)[x + y]^n d\mu_{-1}(y). \tag{2.1}
\]

When \( x = 0 \), \( \mathcal{E}_{n,q,\zeta}(0) = \mathcal{E}_{n,q,\zeta} \) is called the \( n \)-th twisted \( q \)-Euler numbers.

Let \( w_1 \) and \( w_2 \) be odd numbers. Then we have

\[
\begin{align*}
\int_{\mathbb{Z}_p} &\zeta^{w_1y} e^{[w_1]_q y} \left[ w_2x + \frac{w_1}{w_2} j^y \right] q^t d\mu_{-1}(y) \\
&= \lim_{N \to \infty} \sum_{y=0}^{w_2p^{N-1}-1} \zeta^{w_1y} e^{[w_1w_2x + w_2j + w_1]_q y} (-1)^y t^y \\
&= \lim_{N \to \infty} \sum_{i=0}^{w_2-1} \sum_{y=0}^{p^{N-1}-1} \zeta^{w_1(i+w_2y)} e^{[w_1w_2x + w_2j + w_1(i+w_2y)]_q y} (-1)^{i+w_2y} \\
&= \lim_{N \to \infty} \sum_{i=0}^{w_2-1} \sum_{y=0}^{p^{N-1}-1} \zeta^{w_1(i+w_2y)} e^{[w_1w_2x + w_2j + w_1(i+w_2y)]_q y} (-1)^{i+w_2y}.
\end{align*}
\]
From (2.2), we can derive the following equation (2.3):

\[
\sum_{j=0}^{w_1-1} (-1)^j \zeta^{w_2 j} \int_{\mathbb{Z}_p} \zeta^{w_1 y} e^{[w_1]q} \left[ \frac{w_2}{w_1} + \frac{w_1}{w_2} j + y \right] q^{w_1} t d\mu_{-1}(y)
\]

\[
= \lim_{N \to \infty} \sum_{j=0}^{w_1-1} \sum_{i=0}^{w_1-1} \sum_{y=0}^{N-1} (-1)^{i+j} \zeta^{w_2 j} \zeta^{w_1 y} e^{[w_1]q} \left[ w_1 w_2 x + w_1 w_1 y \right] q^{w_1} t (-1)^y
\]

By the same method as (2.3), we have

\[
\sum_{j=0}^{w_2-1} (-1)^j \zeta^{w_1 j} \int_{\mathbb{Z}_p} \zeta^{w_2 y} e^{[w_2]q} \left[ \frac{w_2}{w_1} x + \frac{w_1}{w_2} j + y \right] q^{w_2} t d\mu_{-1}(y)
\]

\[
= \lim_{N \to \infty} \sum_{j=0}^{w_2-1} \sum_{i=0}^{w_2-1} \sum_{y=0}^{N-1} (-1)^{i+j} \zeta^{w_2 j} \zeta^{w_1 y} e^{[w_1]q} \left[ w_1 w_2 x + w_1 w_2 y \right] q^{w_2} t (-1)^y
\]

Therefore, by (2.3) and (2.4), we have the following theorem.

**Theorem 2.1** For \( w_1, w_2 \in \mathbb{N} \) with \( w_1 \equiv 0 \pmod{2} \), \( w_2 \equiv 0 \pmod{2} \), we have

\[
\sum_{j=0}^{w_1-1} (-1)^j \zeta^{w_2 j} \int_{\mathbb{Z}_p} \zeta^{w_1 y} e^{[w_1]q} \left[ \frac{w_2}{w_1} + \frac{w_1}{w_2} j + y \right] q^{w_1} t d\mu_{-1}(y)
\]

\[
= \sum_{j=0}^{w_2-1} (-1)^j \zeta^{w_1 j} \int_{\mathbb{Z}_p} \zeta^{w_2 y} e^{[w_2]q} \left[ \frac{w_1}{w_2} x + \frac{w_2}{w_1} j + y \right] q^{w_2} t d\mu_{-1}(y).
\]

By substituting Taylor series of \( e^{xt} \) into (2.5) and after elementary calculations, we have the following corollary.

**Corollary 2.2** For \( w_1, w_2 \in \mathbb{N} \) with \( w_1 \equiv 0 \pmod{2} \), \( w_2 \equiv 0 \pmod{2} \), we have

\[
[w_1]_q^n \sum_{j=0}^{w_1-1} (-1)^j \zeta^{w_2 j} \int_{\mathbb{Z}_p} \zeta^{w_1 y} \left[ \frac{w_2}{w_1} x + \frac{w_1}{w_2} j + y \right]^n q^{w_1} d\mu_{-1}(y)
\]

\[
= [w_2]_q^n \sum_{j=0}^{w_2-1} (-1)^j \zeta^{w_1 j} \int_{\mathbb{Z}_p} \zeta^{w_2 y} \left[ \frac{w_1}{w_2} x + \frac{w_2}{w_1} j + y \right]^n q^{w_2} d\mu_{-q\equiv 2}(y).
\]
Theorem 2.3 For \(w_1, w_2 \in \mathbb{N}\) with \(w_1 \equiv 0 \pmod{2}\), \(w_2 \equiv 0 \pmod{2}\), we have

\[
[w_1^n q \sum_{j=0}^{w_1-1} (-1)^j \zeta^{w_2 j} \mathcal{E}_{n,q^{w_1},\zeta^{w_1}} \left( w_2 x + \frac{w_2}{w_1} j \right)]
\]

\[
= [w_2^n q \sum_{j=0}^{w_2-1} (-1)^j \zeta^{w_1 j} \mathcal{E}_{n,q^{w_2},\zeta^{w_2}} \left( w_1 x + \frac{w_1}{w_2} j \right)]
\]

By (2.6), we can derive the following equation (2.7):

\[
\int_{\mathbb{Z}_p} \zeta^{w_1 y} \left[ w_2 x + \frac{w_2}{w_1} j + y \right]^n_q d\mu_{-1}(y)
\]

\[
= \sum_{i=0}^{n} \frac{n}{i} \left( \frac{w_2 q}{w_1 q} \right)^i \left[ i \right]_{q^{w_2}} \mathcal{E}_{n-q^{w_2} j} \left( w_2 x + y \right)_{q^{w_2}} d\mu_{-1}(y)
\]

(2.7)

By (2.7), and Theorem 2.3, we have

\[
[w_1^n q \sum_{j=0}^{w_1-1} (-1)^j \zeta^{w_2 j} \int_{\mathbb{Z}_p} \zeta^{w_1 y} \left[ w_2 x + \frac{w_2}{w_1} j + y \right]^n_q d\mu_{-1}(y)]
\]

\[
= \sum_{j=0}^{w_1-1} (-1)^j \zeta^{w_2 j} \sum_{i=0}^{n} \binom{n}{i} \left[ w_2 \right]_{q^{w_1}} \left[ i \right]_{q^{w_2}} \mathcal{E}_{n-i,q^{w_1}\zeta^{w_1}} \left( w_2 x \right)
\]

\[
= \sum_{i=0}^{n} \binom{n}{i} \left[ w_2 \right]_{q^{w_1}} \left[ i \right]_{q^{w_2}} \mathcal{E}_{n-i,q^{w_1}\zeta^{w_1}} \left( w_2 x \right) \sum_{j=0}^{w_1-1} (-1)^j \zeta^{w_2 j} \left[ j \right]_{q^{w_2}}
\]

\[
= \sum_{i=0}^{n} \binom{n}{i} \left[ w_2 \right]_{q^{w_1}} \left[ i \right]_{q^{w_2}} \mathcal{E}_{n-i,q^{w_2}\zeta^{w_2}} \left( w_2 x \right) T_{n,i} \left( w_1, \zeta^{w_2}, q^{w_2} \right)
\]

(2.8)

where

\[
T_{n,i} \left( w_1, \zeta, q \right) = \sum_{j=0}^{w_1-1} (-1)^j \zeta^{j} q^{(n-i)j} \left[ j \right]_{q^i}
\]

By the same method as (2.8), we obtain

\[
[w_2^n q \sum_{j=0}^{w_2-1} (-1)^j \zeta^{w_1 j} \int_{\mathbb{Z}_p} \zeta^{w_2 y} \left[ w_1 x + \frac{w_1}{w_2} j + y \right]^n_{q^{w_2}} d\mu_{-w_2}(y)]
\]

\[
= \sum_{i=0}^{n} \binom{n}{i} \left[ w_1 \right]_{q^{w_2}} \left[ i \right]_{q^{w_1}} \mathcal{E}_{n-i,q^{w_2}\zeta^{w_2}} \left( w_1 x \right) T_{n,i} \left( w_2, \zeta^{w_1}, q^{w_1} \right)
\]

(2.9)

By (2.8) and (2.9), we have the following theorem.
Theorem 2.4 For $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 0 \pmod{2}$, $w_2 \equiv 0 \pmod{2}$, we have

$$
\sum_{i=0}^{n} \binom{n}{i} [w_2]^i [w_1]^{n-i} T_{n,i}(w_1, \zeta^{w_2}, q^{w_2}) E_{n-i, q^{w_1}, \zeta^{w_1}} (w_2 x)
$$

$$
= \sum_{i=0}^{n} \binom{n}{i} [w_1]^i [w_2]^{n-i} T_{n,i}(w_2, \zeta^{w_1}, q^{w_1}) E_{n-i, q^{w_2}, \zeta^{w_2}} (w_1 x).
$$

By (2.5) and Theorem 2.4, we have the following corollary.

Corollary 2.5 For $w_1, w_2 \in \mathbb{N}, n \geq 0$, we have

$$
\sum_{i=0}^{n} \sum_{l=0}^{n-i} \binom{n}{i} \binom{n-i-l}{l} [w_2]^i [w_1]^{n-i-l} T_{n,i}(w_1, \zeta^{w_2}, q^{w_2}) q^{w_1 w_2 x l}[x]^{n-i-l} E_{q^{w_1}, \zeta^{w_1}}
$$

$$
= \sum_{i=0}^{n} \sum_{l=0}^{n-i} \binom{n}{i} \binom{n-i-l}{l} [w_1]^i [w_2]^{n-i-l} T_{n,i}(w_2, \zeta^{w_1}, q^{w_1}) q^{w_1 w_2 x l}[x]^{n-i-l} E_{q^{w_2}, \zeta^{w_2}}.
$$

References


Received: April 11, 2015; Published: May 29, 2015